Submit
Search Options
» Sign up / Log in

English

Academic edition

Skip to: Main content Side column

- Home
- Contact Us

Find out how to access preview-only content

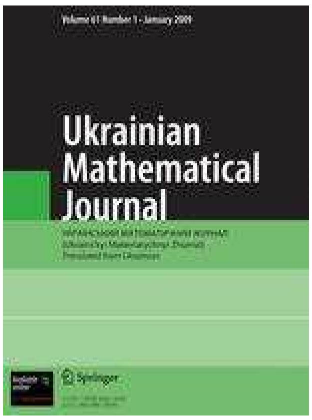
Ukrainian Mathematical Journal
January–February, 1975, Volume 27, Issue 1, pp 56-62

On some properties of the Hadamard products of functions which are regular in the disk

- V. I. Gorbaichuk,
- V. I. Kuz'mich

Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 27, No. 1, pp. 74–81, January–February, 1975.

In conclusion the authors thank R. N. Koval'chuk for suggesting the problem and continued help in the preparation of this article.




ON SOME PROPERTIES OF THE HADAMARD PRODUCTS OF FUNCTIONS WHICH ARE REGULAR IN THE DISK

V. I. Gorbaichuk and V. I. Kuz'mich

Other actions

- Export citation

Supplementary Material (0)

References (17)

About this Article

ON SOME PROPERTIES OF THE HADAMARD PRODUCTS OF FUNCTIONS WHICH ARE REGULAR IN THE DISK

V. I. Gorbaichuk and V. I. Kuz'mich

UDC 517.544

1. Let us denote by H_p, $p > 0$, the class of functions f(z) which are regular in the disk $D = \{z : |z| < 1\}$ and are such that the integral $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$, where $0 < r < 1$, is bounded for each of them. We know [1, p. 389] that every function $f(z) \in H_p$, $p > 0$, has definite limiting values along nontangential paths almost everywhere on $\gamma = \{z : |z| = 1\}$ forming the boundary function $f(e^{i\theta})$. It follows from Fatou's theorem [2, p. 66] that $f(e^{\theta i}) \in L_p$ in $(0, 2\pi)$, i.e., $\left\{\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right\}^{1/p} < \infty$. In what follows we shall denote by h_p, $p > 0$, the class of functions $u(r, \theta)$ which are harmonic in the disk D and are such that the integral $\int_0^{2\pi} |u(r, \theta)|^p d\theta$, where $0 < r < 1$, is bounded for each of them.

Let the functions f(z) and g(z) which are regular in the disk D be defined by power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{1}$$

Let us form the function

$$F(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \tag{2}$$

which is called the Hadamard product of the functions f(z) and g(z) (see [3], and also [4, pp. 37 and 38]).

In [4] are set forth deep results about the study of the properties of the function F(z) clustering around Hadamard's theorem on the product of singularities (in which is also given an extensive bibliography on these problems).

V. Daiovitch [5, 6], N. A. Davydov [7], Qiu Hua-Ji [8] (see [9]) and J. P. Milaszewicz [10] have studied the structural properties of the function F(z) in relation to the structural properties of the functions f(z) and g(z). The following results are characteristic of this group of problems.

<u>THEOREM 1.</u> If $f(z) \in H_p$ and $\mathrm{Re}\, g(z) \in h_q$ in D, where $p > 1$, $q > 1$, and $p^{-1} + q^{-1} = 1$, then F(z) is bounded in D and has an angular boundary value almost everywhere on γ.

2. If $f(z) \in H_p$ and $\mathrm{Re}\, g(z) \in h_1$, then $F(z) \in H_p$, $1 \le p \le \infty$.

The first part of the theorem is due to V. Daiovitch [6] and was reproved in [8], and the second part has been proved by Qiu Hua-Ji [8] and has been obtained as a consequence of more general reasonings by J. P. Milaszewicz [10].

The Poisson integral has been used in [6] and [10] for obtaining the above-stated results. As remarked in [11], these results can be obtained with the help of the Parseval contour integral

$$F(z) = \frac{1}{2\pi i} \int_C f(t)\, g\left(\frac{z}{t}\right) \frac{dt}{t}. \tag{3}$$

Luts Pedagogical Institute. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 27, No. 1, pp. 74–81, January–February, 1975. Original article submitted June 30, 1973.

where C is a closed rectifiable contour lying in the domain of regularity of the function being integrated.

We have verified that the first part of the theorem can be proved by this method (see below the proof of the necessity of the conditions of Theorem 4), and the second part has been successfully proved only for $p > 1$ since it is necessary to use a theorem due to M. Riesz [1, p. 380] which is valid only for $p > 1$.

Qiu Hua-Ji [8] uses the following integral representation:

$$F(z) = \frac{1}{\pi}\int_0^{2\pi} u(r, \varphi) f(r, e^{i(\theta-\varphi)})\, d\varphi + c_0, \tag{4}$$

where $z = \rho e^{i\theta}$, $\rho = r^2$, $0 < \rho < 1$, $g(re^{i\varphi}) = u(r, \varphi) + iv(r, \varphi)$, and c_0 is a constant.

For the study of the other (boundary) properties of the function $F(z)$ Qiu Hua-Ji [8] uses the equation

$$\lim_{\rho\to 1-0} F(\rho, e^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} u(1, \varphi) f(1, e^{i(\theta-\varphi)}) d\varphi + c_0 \tag{5}$$

in the premises of the theorem (the part 1 or 2). But, as observed validly by G. Ts. Tumarkin [12], the integral on the right-hand side of Eq. (5) may turn out to be divergent under these conditions.

In what follows we shall use the integral representation (3). For this, on putting $C = \{t : |t| = \rho,\ |z| < \rho < 1\}$, we get

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\tau})\, g\left(\frac{z}{\rho e^{i\tau}}\right) d\tau. \tag{6}$$

For the study of the boundary properties of the function $F(z)$ let us assume that the integral $(2\pi)^{-1}\int_0^{2\pi} f(1, e^{i\tau}) g(z/\rho e^{i\tau}) d\tau$ and the integral on the right-hand side of (5) converge. Sometimes these integrals converge by virtue of the conditions imposed on the functions being integrated and Hölder's inequality (this will, e.g., be the case when $u(1, \varphi) \in L_p$, $p > 1$, $f(1, e^{i(\theta-\varphi)}) \in L_q$, $q > 1$, and $p^{-1} + q^{-1} = 1$). But in the cases where the convergence does not follow from such reasonings, the assumption of the convergence of the above-mentioned integrals extends the possibility of further considerations since, e.g., the function $u(1, \varphi) f(1, e^{i(\theta-\varphi)})$ may be summable under the condition that one of the functions $u(1, \varphi)$ and $f(1, e^{i(\theta-\varphi)})$ is nonsummable over the same interval.

In such a case, according to a theorem of F. Riesz [1, p. 390], it is possible to pass to limit under the sign of integration and the following equations hold good:

$$F(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\tau})\, g(e^{i(\theta-\tau)})\, d\tau, \tag{7}$$

$$F(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(\varphi) f(e^{i(\theta-\varphi)})\, d\varphi + c_0 \tag{8}$$

(in the relation (8) we have put, for brevity, $u(1, \varphi) = u(\varphi)$ and $f(1, e^{i(\theta-\varphi)}) = f(e^{i(\theta-\varphi)})$.

Under these initial assumptions and notation we shall study some differential-difference properties of the Hadamard product in the following section.

3. Let the function $\varphi(z) \in H_p$, $p > 1$, in the disk D and $\varphi(\theta)$ be the angular boundary value of the function $\varphi(z)$ almost everywhere on γ. Then, as we know, $\varphi(\theta) \in L_{p[0,2\pi]}$. The integral modulus of continuity of k-th order $\omega_k^{(p)}(\varphi; t)$ of the function $\varphi(\theta)$ is defined by the relation

$$\omega_k^{(p)}(\varphi; t) = \sup_{0<h\leq t}\left\{\int_0^{2\pi} |\Delta_h^k \varphi(\theta)|^p d\theta\right\}^{\frac{1}{p}},$$

where $\Delta_h^k \varphi(\theta) = \sum_{\nu=0}^{k} (-1)^{k-\nu}\binom{k}{\nu}\varphi(\theta + kh)$ is the k-th order difference of the function $\varphi(\theta)$ given on γ.