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V I Kuz'mich and L V Kuzmich

Kherson State University, 27 Universytetska Str., Kherson, 73003, Ukraine

E-mail: vikuzmichksu@gmail.com

Abstract. The paper deals with issues of the metric geometry basics. In particular, the concept of rectilinear placement of points is considered, based on the axioms of the distance between two points of metric space. This approach allows forming a modern view of the property of straightness in the pupils. This paper analyzes the content of existing mathematics textbooks for general educational institutions to acquaintance of pupils with the elements of metric geometry. The first part of the paper provides information about the rectilinear placement of points; it can be used in Geometry lessons in the 7^{th} – 9^{th} grades. Set of linear functions are considered as examples of points of metric space. The similar work was done in the second part of the work for geometric material of the 10^{th} – 11^{th} grades. In addition, some simple examples of metric spaces that may be accessible to pupils of the relevant classes are discussed. The purpose of the work is gradually introduction of pupils to the elements of non-Euclidean geometries, to form a generalized notion of the distance between the points and rectilinear of their placement. The work can be used for Mathematics teaching at school and for retraining of teachers of Mathematics.

1. Introduction

Mathematics is a fundamental science not only for engineering disciplines, but also for other disciplines. But in most cases, it is recognized "as too difficult" and students refuse to study it in favor of other sciences [1, 2, 3].

One of the advantages of mathematical education is the ability to build abstract models [4, 5]. These models are a set of graphical constructions and algebraic formulas that describe phenomena and events, and are used in forecasting. In particular, in art, the application of the properties of non-Euclidean elements allows to obtain 2D and 3D interesting results, such as tessellation, Escher's hyperbolic art and some paradoxes on Magritte's work [6, 7]. The concept of metric space in non-Euclidean geometry is the learning basis in navigation, satellite communications, phenomena that take place over very long distances [8]. At most issues study, clarity is no longer available, only analytical researches are possible, they are considered difficult. In order to interest students in the study of mathematical disciplines, it is necessary to stimulate their early motivation for analytical constructions [9, 10, 11].

Appropriate teacher training should be conducted. The curriculum of the school mathematics in the last half-century has undergone significant changes due to the widespread use of the latest mathematical research methods. In Geometry, the part of educational material is studied using vectors and the boundary transition, in Algebra and in the basis of Mathematical analysis the elements of combinatorics, probability theory and statistics, derivative of function, indefinite and definite integral are studied.

The rapid development of non-Euclidean geometries, and their practical application in modern researches and technologies, raises the question of the need for students to get acquainted with the basic concepts and provisions of these geometries in school mathematics. It will allow students to form a critical attitude to basic mathematical concepts and postulates, and will help develop the ability to analyze them and apply them to build mathematical models of various phenomena and processes. The simplest way to achieve this goal, in our opinion, is elements of metric geometry use, as it is the closest to the classical Euclid geometry. It should be noted the simple analytical transformations in establishing the elementary facts of metric geometry, because they are based on clear axioms of the distance between the points of metric space. According to the authors of the metric geometry course, "... metric geometry remains; perhaps, it is one of the most "elementary" mathematical methods".

This paper will show how the means of metric geometry can be applied to the formation of the concept of the distance between two points and the concept of straightness, based on the school mathematics course. It will help in teaching to motivate students to study the exact sciences and further science and mathematics research, including STEM sciences.

2. Formation of notions of distance and straightness by means of metric geometry in 7th- 9th grades

As a rule, future mathematics teachers will be introduced to metric spaces in the mathematical analysis course, at studying the functions of several variables. Introducing the concept of *n*-dimensional of Euclidean space the generalized notion of the distance between two points of this space is presented. It generalizes the concept of the distance between two points on a numerical axis, the distance between two points on a coordinate plane, and the distance between two points of coordinate space of three-dimensional, with which students are familiar by the school mathematics. A general definition of metric space and a detailed study of specific metric spaces are offered in Functional analysis course. Here are some basic definitions of the metric spaces.

Definition 1. *A metric space is a totality of a nonempty set* X *of elements of any nature and a single-valued real non-negative function* $\rho(x;y)$ *defined for any elements* x *and* y *from* X, *and satisfying the following conditions:*

1) $\rho(x;y)=0$ *if and only if* $x=y$*;*

2) $\rho(x;y)=\rho(y;x)$ *(axiom of symmetry);*

3) for any three elements x, y *and* z, *the inequality is satisfied*

$\rho(x;y)\leq\rho(x;z)+\rho(z;y)$ *(axiom of a triangle)*

(see, the example, [12, p. 102]).

The elements of the set X are called points of the metric space, the function ρ is the metric of X space, and the numerical value of the function $\rho(x;y)$ is the distance between the elements (points) x and y. The metric space X with ρ metric is denoted by $(X;\rho)$. Terms 1), 2) and 3) of Definition 1 is also called distance axioms.

It is impractical to introduce the notion of metric and metric space in the form given in Definition 1, because the notion of function of two variables, or the functional, since points of space X can be any is used.

Note that in Definition 1, the elements of the set X can be of any nature. Euclid described the point as follows: "The point has no parts" [13, p. 11], Heron "the point has no magnitude (length)" [13, p. 224]. It is consistent with the description of the point in the school textbooks: "The point is the simplest geometric figure. It is the only figure that cannot be divided into parts" [14, p. 12]; "The point has neither length nor width, its shape we cannot determine" [15, p. 15]. Sometimes the point is described by its graphical representation: "If a well pointed pencil is pressed on a piece of paper, there will be a trace that gives an idea of the point" [16, p. 9]; "The idea of a point can be obtained by pressing a well-sharpened pencil on a piece of paper or a well-sharpened piece of chalk on a school board" [17, p. 6], or simply: "The simplest geometric figure is a point" [18, p. 6].

In our view, the concept of a point could be described in more detail, indicating that any object can be considered as a point when its structure, form, properties, etc. are not used. For example, when it is necessary to calculate the number of buildings in a certain area and to set distances between them, without considering the size, shape, number of floors and apartments of these buildings, although each of the buildings has these characteristics and can be used in the future. At acquainting with the concept of the set, it should be emphasized that it is a set of objects (elements) united by a certain characteristic: the set of *pupils of one class*, the set of *even numbers*, etc. All the elements (points) of the set are equal among themselves, however at dealing with them it is necessary to check this points for the fulfillment of the characteristic by what they belong to this set, and you can also use this characteristic for the operations with points of this set. This concept of the point is broader than the concept of the point in Euclid's geometry, but it more accurately reflects the modern view at the point as an element of the set. The proposed point description is fully consistent with Definition 1, and prepares students for a generalized perception of point concepts and distances between points in specific metric spaces.

The first acquaintance to the concept of the distance between two points at the definition level occurs in the seventh grade at learning the basic geometric concepts: "The length of the segment AB is called the distance between points A and B. If points A and B are coincided, then the distance between them is zero" [14, p. 17]; "The distance between two points is the length of the segment with the ends at these points" [15, p. 18]; "The length of the segment AB is also called the distance between points A and B" [16, p. 17]. At this stage of studying mathematics, in our opinion, it is too early to talk about other definitions of the distance between points, although it is possible to draw pupils attention to the fact that at driving around the city, the shortest distance they have to overcome between two objects is not always measured by the length of the segment that connects these objects and may be exceeded them. Moreover, there may be several such paths. It may be the first example of the ambiguity of the notion of the distance between two points. It will help to train students for the perception of a further triangle inequality (Condition 3) of Definition 1. Pupils will also be introduced to this inequality in the seventh grade: "Each side of the triangle less then sum of the other two sides…" [14, p. 113; 15, p. 109; 16, p. 74; 17, p. 115; 18, pp. 108-109].

Due to the triangle's inequality, the characteristic property of three points belonging to one straight line is deduced, and the notion "point B is contained between points A and C", which means the "rectilinear placement" of these points A, B, C: "if the equality $AB = AC + CB$ is satisfied for the three points A, B and C, then the point C is the inner point of the segment AB" [14, p. 114]; "If the equality $AB + BC = AC$ is satisfied for the three points A, B, C, then these points lie on one line and point B is between points A and C" [15, p. 109]; "... if point C lies between points A and B ..., then the following proportions are correct: $AB = BC + CA$, $BC < CA + AB$, $CA < AB + BC$" [18, p. 109]. These facts are in full agreement with the axioms of placement proposed by V. F. Kagan in constructing a straight line theory.

Purposeful and detailed introduction of students to the elements of metric geometry should begin, in our opinion, in the ninth grade. In Geometry of ninth-grade, the basic material for the elements metric geometry as trigonometry elements, cosine theorem, sine theorem, triangle's inequality, triangles solving, Cartesian coordinates on plane, scalar product of vectors is studied. In addition, the parallel study in Algebra the properties of functions gives the opportunity to begin the study of specific metric spaces, such as the space of linear (or quadratic) functions on a segment. An experienced teacher, according to the time available to study mathematics, will easily be able to divide some facts from metric geometry into those that can be learned in class and those that need to be learned in the after school.

Now let's consider the actual material that is being offered for study. First, we formulate a somewhat simplified, but more voluminous, definition of the metric space and the distance between its points. This definition uses the notion of the set and its elements studied in the eighth grade [19, p. 24].

Definition 2. *Nonempty set X of elements of any nature will call a metric space if each pair $(x; y)$ of different elements X, by some rule ρ, is matched by a single real non-negative number $\rho(x; y)$, which is called the distance between the elements x and y, and which satisfies the conditions:*

1) for any two different elements x and y, the distance between the elements x and y is equal to the distance between the elements y and x, the equality $\rho(x; y) = \rho(y; x)$ (symmetry *conditions) is satisfied;*

2) for any three different elements x, y, z, the distance between the elements x and y is not greater than the sum of the distances between the elements x and z and between the elements z and y, the inequality $\rho(x;y) \leq \rho(x;z) + \rho(z;y)$ (triangle inequality) is satisfied.

If the conditions of Definition 2 are satisfied, then the elements of the set X will be called points of metric space, the rule ρ – metric of space. The metric space X with metric ρ will be denoted (X, ρ).

It should be noted that this definition is reminiscent of the function which is being studied in Algebra in the ninth grade [20, p. 11]. However, there are several significant differences: the elements of set can be not only numbers, the number is matched to the two elements of the set, and this number should only be positive. In addition, the symmetry condition and triangle inequality for all pairs of elements of the set should be checked.

The Definition 2 of metric space, in the form as it is written, should be given in the senior classes, and in the ninth grades it is advisable to give it (as well as the function's definition) in a descriptive form, using a sufficient number of examples. In this case, it is possible to divide the formulation of conditions 1) and 2) definitions into verbal and analytical forms.

Here are some of the simplest examples of metric spaces available for easy learning by pupils.

Example 1. The simplest example of the metric space is the set of all points of the numerical axis. This space is called a one-dimensional arithmetic Euclidean space, and denote R^1. As it is known [21, p. 82], the distance between two points x and y of the numerical axis is found as the absolute value (modulus) of the difference of the corresponding numbers x and y: $\rho(x;y) = |x - y|$.

This value is always positive for different values of x and y, it follows from the definition of the number of module. The symmetry condition follows from the equalities:

$$\rho(x;y) = |x - y| = |-(x - y)| = |y - x| = \rho(y;x).$$

Checking the inequality of the triangle is always the most difficult because it is difficult enough to prove inequalities. For this case, the inequality of a triangle looks like:

$$\rho(x;y) = |x - y| \leq |x - z| + |z - y| = \rho(x;z) + \rho(z;y). \quad (1)$$

At its prooving, the inequality for the modulus of the sum of two numbers is used [20, p. 60]:

$$|a + b| \leq |a| + |b|. \quad (2)$$

If in inequality (2) we put: $a = x - z$, $b = z - y$, then we get inequality (1).

Since all the conditions of Definition 2 are fulfilled, then the considered space R^1 is metric.

Inequality (1) can be proved in the ninth grade. However, using a number line, you can try to prove it even in the seventh grade. To do this, we have to consider six different possible cases of placement of x, y, z points on the numerical axis. A large number of analytical transformations can be offset by graphical representation of points on the numerical axis; it will make easier to understand such transformations.

For example, let the points x, y, z are placed on a number line in the following order: $x < z < y$. Then we will have:

$$\rho(x;y) = |x - y| = -(x - y);$$

$$\rho(x;z) = |x - z| = -(x - z);$$

$$\rho(z;y) = |z - y| = -(z - y).$$

Adding the last two equalities we will have:

$$\rho(x;z) + \rho(z;y) = -(x - z) - (z - y) = -x + z - z + y = -(x - y) = \rho(x;y).$$

Other placement cases of x, y, z points are observed similarly.

Example 2. An example of a metric space is the set of points of a coordinate plane. It is studied in detail in Geometry for ninth grade [21, p. 81; 23, p. 8; 24, p. 6; 25, p. 6; 26, p. 9]. This space is called a two-dimensional arithmetic Euclidean space, and is denoted R^2.

Distance between two points $M_1(x_1; y_1)$ and$M_2(x_2; y_2)$ of space R^2 is defined as the length of the segment M_1M_2, which is presented by the formula [21, p. 82; 23, p. 8; 24, p. 24; 25, p. 32; 26, p. 12]:

$$M_1M_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

For two different points of the coordinate plane, this distance is positive and has the property of symmetry:

$$\rho(M_1; M_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \rho(M_2; M_1).$$

Proving the inequality of a triangle is a bit more complicated. However, the using the Cauchy-Bunyakovsky inequality is facilitated [20, p. 209]. For arbitrary four values: a_1, a_2, b_1, b_2 it looks like:

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2).$$

It is easy to prove this inequality by raise to the second degree in the left part of it and multiplying the expressions in brackets in the right part of the inequality. Since both parts of the inequality are non-negative, extracting from them the square root we get the inequality:

$$a_1b_1 + a_2b_2 \leq \sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}. \tag{3}$$

The triangle inequality for three points$M_1(x_1; y_1)$, $M_2(x_2; y_2)$, $M_3(x_3; y_3)$ has the form:

$$\rho(M_1; M_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}.$$
$$+\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = \rho(M_1; M_3) + \rho(M_3; M_2).$$

Denote by: $x_1 - x_3 = a_1,\ x_3 - x_2 = b_1,\ y_1 - y_3 = a_2,\ y_3 - y_2 = b_2$. These values will be substituted in the triangle inequality:

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}.$$

Since both parts of the inequality are non-negative, we raise them to the square and we get the inequality:

$$(a_1 + b_1)^2 + (a_2 + b_2)^2 \leq a_1^2 + a_2^2 + 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} + b_1^2 + b_2^2. \tag{4}$$

Let's transform the left part of inequality (4) and use inequality (3):

$$(a_1 + b_1)^2 + (a_2 + b_2)^2 = a_1^2 + 2a_1b_1 + b_1^2 + a_2^2 + 2a_2b_2 + b_2^2 =$$
$$= a_1^2 + a_2^2 + 2(a_1b_1 + a_2b_2) + b_1^2 + b_2^2 \leq a_1^2 + a_2^2 + 2\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)} + b_1^2 + b_2^2.$$

The inequality (4) is presented. Therefore, the inequality of the triangle is satisfied. Since all the conditions of Definition 2 are fulfilled, then the considered space R^2 is metric.

The other metric can be selected on the coordinate plane, and thus the points of the plane will form a different metric space other than the space R^2.

Example 3. Consider as the distance between the points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ of the coordinate plane the number:

$$\rho(M_1; M_2) = |x_1 - x_2| + |y_1 - y_2|.$$

This number for two different points is positive. The symmetry condition is obvious due to the properties of the number module. The triangle inequality for three points $M_1(x_1; y_1)$, $M_2(x_2; y_2)$, $M_3(x_3; y_3)$ has the form:

$$\rho(M_1; M_2) = |x_1 - x_2| + |y_1 - y_2| \leq$$
$$\leq (|x_1 - x_3| + |y_1 - y_3|) + (|x_3 - x_2| + |y_3 - y_2| = \rho(M_1; M_3) + \rho(M_3; M_2).$$

The correct of this inequality is obvious, since each of the modules on the left side of the inequality does not exceed the sum of the modules of its terms.

All conditions of Definition 2 are fulfilled, so the considered space is metric. This space is denoted by R_1^2.

The space R_1^2 is interesting because its essence is easy to explain even to the seventh grade pupils. By this metric, on the coordinate plane, the smallest distance between the points M_1 and M_2 can be overcome by walking parallel to the coordinate axes (along the legs of a right triangle for which the segment M_1M_2 is a hypotenuse). A similar situation occurs in a city with a rectangular street layout. This is one example where the concept of the distance between points does not coincide with the classic, as the length of the segment connecting these points.

Example 4. Consider as the distance between the points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ of the coordinate plane the number:

$$\rho(M_1; M_2) = max\{|x_1 - x_2|;\ |y_1 - y_2|\}.$$

This number for two different points is positive. The symmetry condition is obvious due to the properties of the number module. The triangle inequality for three points $M_1(x_1; y_1)$, $M_2(x_2; y_2)$, $M_1(x_1; y_1)$ has the form:

$$\rho(M_1; M_2) = max\{|x_1 - x_2|;\ |y_1 - y_2|\} \leq max\{|x_1 - x_3|;\ |y_1 - y_3|\} + \\ + max\{|x_3 - x_2|;\ |y_3 - y_2|\} = \rho(M_1; M_3) + \rho(M_3; M_2).$$

Using the inequality for the modulus of the sum of two numbers, we obtain:

$$|x_1 - x_2| = |(x_1 - x_3) + (x_3 - x_2)| \leq |x_1 - x_3| + |x_3 - x_2| \leq \\ \leq \max\{|x_1 - x_3|; |y_1 - y_3|\} + max\ \{|x_3 - x_2|; |y_3 - y_2|\}.$$

Similarly, we get inequality:

$$|y_1 - y_2| = |(y_1 - y_3) + (y_3 - y_2)| \leq |y_1 - y_3| + |y_3 - y_2| \leq \\ \leq \max\{|x_1 - x_3|; |y_1 - y_3|\} + max\ \{|x_3 - x_2|; |y_3 - y_2|\}.$$

Comparing both obtained inequalities, we finally get:

$$\rho(M_1; M_2) = max\{|x_1 - x_2|;\ |y_1 - y_2|\} \leq \\ \leq ma\,x\{|x_1 - x_3|; |y_1 - y_3|\} + ma\,x\{|x_3 - x_2|; |y_3 - y_2|\} = \rho(M_1; M_3) + \rho(M_3; M_2).$$

Thus, all conditions of Definition 2 are fulfilled, so the considered space is metric and is denoted by R_0^2. Sometimes such a space is more convenient than the space R^2. The space R_0^2 can also be an example of space in which the distance between points is not always the length of the segment connecting these points.

Example 5. Consider in the space R_0^2 four points: $M_1(0; 1)$, $M_2(0; -1)$, $M_3(-1; 0)$, $M_4(1; 0)$. Let's find by the metric of space the distance between these points:

$$\rho(M_1; M_2) = 2, \rho(M_1; M_3) = 1, \rho(M_1; M_4) = 1, \rho(M_2; M_3) = 1, \rho(M_2; M_4) = 1, \rho(M_3; M_4) = 2.$$

It is necessary to pay attention to the equalities that are fulfilled:

$$\rho(M_1; M_2) = \rho(M_1; M_3) + \rho(M_2; M_3) = 1 + 1 = 2;$$

$$\rho(M_1; M_2) = \rho(M_1; M_4) + \rho(M_2; M_4) = 1 + 1 = 2;$$

$$\rho(M_3; M_4) = \rho(M_1; M_3) + \rho(M_1; M_4) = 1 + 1 = 2;$$

$$\rho(M_3; M_4) = \rho(M_2; M_3) + \rho(M_2; M_4) = 1 + 1 = 2.$$

Geometrically, on the coordinate plane, the pointsM_1, M_2, M_3, M_4 are the vertices of a square, its side length is $\sqrt{2}$. In Euclid's geometry, the length of the diagonal of the square is less than the sum of the lengths of its two sides, but in this example they are equal. Moreover, in Euclid's geometry there are all three points involved in equality must lie on one straight line.

This example clearly demonstrates the difference of the concepts of distance between points of the same set at different attributes. In addition, this example points to the ambiguity (relativity) of the concept of rectilinear placement of points of metric space. In particular, in his axiomatic, D. Hilbert, in contrast to Euclid, did not give specific definitions of basic geometric concepts: point, line, plane, but he only described their properties because of the correlation between them [27, pp. 3-4].

Let's consider the notion of rectilinear placement of points of metric space. It is a special case of Definition 2, at the inequality of a triangle becomes equality.

Definition 3. *Let's say that the points x, y, z of the metric space (X,ρ) are rectilinear placement in this space if the equality is satisfied:*

$$\rho(x;y)=\rho(x;z)+\rho(z;y). \tag{5}$$

At equality (5) solving, it is natural to say that the point z "lies between" points x and y, or to call it "internal" for points x, y, z. At the same time, the point x (point y) can be said to be "beyond" points y and z (points x and z), or "external" for points x, y, z (compare [14, p. 16]).

For the pupils it may be noted that equality (5) must be executed for some two points from three given points (for example, for points x and y). For other pairs of points the equality $\rho(x;z)=\rho(x;y)-\rho(z;y)$ will be executed, or the equality $\rho(z;y)=\rho(x;y)-\rho(x;z)$, they can also indicate the rectilinear placement of the points x, y, z.

Let's define a rectilinear placement of the set of points of the metric space as the rectilinear placement of any three points of this set.

Definition 4. *Let's say that the set of points of a metric space is rectilinear placement if any three points of that set are rectilinear placement.*

Definitions 3 and 4 make it possible to study the individual properties of rectilinear placement without using a straight line definition and without introducing straight line axioms. They can be used to construct the rectilinear placement sets of points in the arbitrary metric space. The properties of such sets will largely depend on the metric in the corresponding space.

Now let's look at some examples of rectilinear placement of the points in different metric spaces.

Example 6. The simplest example of the set with a rectilinear placement is the space R^1. Indeed, from the properties of the set of real (natural, integer, rational) numbers, it follows that one out of three different numbers x, y, z will be the smallest, the second – the largest, and the third – the intermediate. For example, a double inequality is executed: $x<z<y$. As in Example 1, by the space metric R^1, we find the distances:

$$\rho(x;y)=|x-y|=y-x, \rho(x;z)=|x-z|=z-x, \rho(z;y)=|z-y|=y-z.$$

Since equality:

$$\rho(x;y)=y-x=(z-x)+(y-z)=\rho(x;z)+\rho(z;y),$$

is executed, then by Definition 3 the points, y, z are rectilinearly placed in the space R^1. These points were arbitrary, so Definition 4 is executed and the entire space R^1 is rectilinearly placed.

Example 7. Here is a more complicated example of a rectilinearly placed set. To do this, we consider the set of linear functions $y=kx$ given on the segment $x\in[0;1]$. The graphs of these functions are straight lines passing through the beginning of coordinate. The pupils get acquainted with the properties of functions quite thoroughly in the ninth grade [20, p. 24; 22, p. 65; 28, p. 72; 29, p. 68; 30, p. 73]. However, with some elementary functions and their simplest properties, in particular with a linear function, acquaintance begins in the seventh grade [31, p. 137; 32, p. 139; 33, p. 103; 34, p. 96; 35, p. 130; 36, p. 141]. The two functions defined at a certain interval will be considered different if they have different values at least at one point in that interval. Provided that these functions are continuous on a numerical gap, they will have different values on the some numerical gap.

Let's introduce a metric in this set by choosing the distance between its two different elements $y=k_1x$ and $y=k_2x$ number:

$$\rho(k_1x\,;k_2x)=max_{x\in[0;1]}|k_1x-k_2x|.$$

Let's show that with this choice of the distance between the elements, the set of functions $y = kx$ is the metric space. In the future, for convenience, we will use the notation:

$$k_i x = y_i,\ \rho(k_i x\,; k_j x) = \rho(y_i\,; y_j) = \rho_{ij}\ (i, j = 1,2, \dots).$$

For two different functions, the distance ρ_{12} is positive due to the definition of the number module. If we assume that this distance is zero, then at each point in the segment $[0; 1]$ the values of both functions must be the same, that is, the functions must be coincided.

From the property of modulus of the number the distance symmetry property follows:

$$\rho(y_1\,; y_2) = max_{x\in[0;1]}|k_1 x - k_2 x| = max_{x\in[0;1]}|k_2 x - k_1 x| = \rho(y_2\,; y_1).$$

Consider on the segment $[0; 1]$ three functions: $y_1 = k_1 x$, $y_2 = k_2 x$, $y_3 = k_3 x$, where k_1, k_2, k_3 are different numbers. For example, let's the inequalities $k_1 < k_2 < k_3$ are executed. Let's find the distances between functions:

$$\rho_{12} = max_{x\in[0;1]}|k_1 x - k_2 x| = max_{x\in[0;1]}|k_1 - k_2||x| = k_2 - k_1,$$

$$\rho_{13} = max_{x\in[0;1]}|k_1 x - k_3 x| = max_{x\in[0;1]}|k_1 - k_3||x| = k_3 - k_1,$$

$$\rho_{23} = max_{x\in[0;1]}|k_2 x - k_3 x| = max_{x\in[0;1]}|k_2 - k_3||x| = k_3 - k_2.$$

From the obtained values, the equality follows:

$$\rho_{13} = k_3 - k_1 = (k_2 - k_1) + (k_3 - k_2) = \rho_{12} + \rho_{23}. \tag{6}$$

Thus, the triangle inequality for the points y_1, y_2, y_3 is satisfied, and the chosen distance is a metric. Consequently the set of functions $y = kx$ on the segment $x \in [0; 1]$ is a metric space.

Since for points y_1, y_2, y_3, equality (5) is executed, and the points are choose arbitrarily, then from the equality (6), by Definition 4, the rectilinear placement of the whole set of functions is followed.

Example 8. The space R^1 considered in Example 1, as well as the space R_1^2 considered in Example 3, are partial cases of a more general metric space – R_1^n. This space consists of ordered groups n real numbers: $x(x_1, \dots, x_n)$. The distance between two points $x(x_1, \dots, x_n)$ and $y(y_1, \dots, y_n)$ of this space is given by the formula:

$$\rho(x\,; y) = \textstyle\sum_{k=1}^{n}|x_k - y_k|. \tag{7}$$

The fulfillment of all axioms of the distance for this metric is executed in the same way as in Example 3.

Consider the set P of points of space R_1^n such that for arbitrary three points $x(x_1, \dots, x_n)$, $y(y_1, \dots, y_n)$ and $z(z_1, \dots, z_n)$ of this set the inequalities are executed: $x_k \le y_k \le z_k$ for of all values of $k = 1,2, \dots, n$. Such a set is rectilinear placement in the space R_1^n. Indeed, using equality (7) we have:

$$\begin{aligned}\rho(x\,; z) &= \sum_{k=1}^{n}|x_k - z_k| = \sum_{k=1}^{n}(z_k - x_k) = \sum_{k=1}^{n}\big((z_k - y_k) + (y_k - x_k)\big) = \\ &= \sum_{k=1}^{n}(z_k - y_k) + \sum_{k=1}^{n}(y_k - x_k) = \sum_{k=1}^{n}|z_k - y_k| + \sum_{k=1}^{n}|y_k - x_k| = \\ &= \rho(z\,; y) + \rho(y\,; x) = \rho(x\,; y) + \rho(y\,; z)\end{aligned}$$

Therefore, for points x, y, z the equality (5) is executed, so by Definition 3 they are rectilinear placement in the space R_1^n. Since we have taken these points arbitrarily from the set P, then by Definition 4 this set is rectilinear placement in the space R_1^n.

Rectilinear placement of the points in Examples 6-8 may be intuitively associated with rectilinear placement of the points in Euclid's geometry, however, this is not always true. The following example demonstrates this peculiarity of rectilinear placement of points in metric space.

Example 9. In Example 5, we found that for any three points, of the considered four M_1, M_2, M_3, M_4 in this example, the equality (5) is executed, and therefore every three points, by Definition 3, are rectilinear placement, and hence, by Definition 4, all four points are rectilinear placement in the space R_0^2. This result is somewhat different from the intuitive perception of the notion of a straight line in Euclid's geometry, since these four points are the vertices of the square, as noted above.

We considered some basic concepts of metric geometry and some of the simplest metric spaces which you can introduce middle-class pupils. The teacher can choose the level of validity of the results – intuitive, graphic or rigorous analytical. In the senior classes, more complex metric spaces that require concepts of continuity, differentiation, and integration of function can be considered.

3. Formation of notions of distance and straightness by means of metric geometry in 10th -11th grades

Opportunities for the use of elements of metric geometry in the senior grades are increasing. This is due to a more detailed and thorough study of the properties of functions, in particular on the basis of differential and integral calculus. For example, familiarity with the properties of functions continuous on a segment makes it possible to consider the corresponding metric space. It should be noted that in the senior grades, such material should be considered only if the study of mathematics is at an advanced level.

After acquaintance with continuous functions, their properties and the second Weierstrass theorem [37, p. 318] on the existence of the largest and smallest values of the function continuous on a segment, we can consider the metric space $C_{[a;b]}$ – the set of functions continuous on a segment $[a;b]$ for which the distance between the functions $f(x)$ and $g(x)$ of the set is determined by the formula:

$$\rho(f\,;g) = max_{x\in[a;b]}|f(x) - g(x)|. \tag{8}$$

At this choice of metric, the set of functions becomes a metric space because all axioms of distance are fulfilled (see Example 7).

First, the right side of the equality is always non-negative. If the functions f (x) and g (x) on the segment [a; b] at least in one point have different meanings (these functions are different), then the right side of the equality (8) is positive. The symmetry property is executed due to the property of the number module: $|f(x) - g(x)| = |g(x) - f(x)|$. The execution of the triangle inequality is followed from the obvious inequalities:

$$|f(x) - g(x)| = \big|\big(f(x) - h(x)\big) + \big(h(x) - f(x)\big)\big| \le |f(x) - h(x)| +$$
$$+|h(x) - g(x)| \le max_{x\in[a;b]}(|f(x) - h(x)| + |h(x) - g(x)|) \le max_{x\in[a;b]}|f(x) - h(x)| +$$
$$+max_{x\in[a;b]}|h(x) - g(x)| = \rho(f;h) + \rho(h;g).$$

Since these inequalities are satisfied for an arbitrary value of *x* from the segment [*a*; *b*], we finally obtain the inequality:

$$\rho(f\,;g) = max_{x\in[a;b]}|f(x) - g(x)| \le$$
$$\le max_{x\in[a;b]}|f(x) - h(x)| + max_{x\in[a;b]}|h(x) - g(x)| = \rho(f;h) + \rho(h;g).$$

It should be noted that all the maxima that are part of the inequalities are due to the continuity on a segment $[a;b]$ of the corresponding functions [37, pp. 305-308].

Since all the basic elementary functions studied in a school mathematics course are continuous in their fields of definition, then metric (8) can be used to study the same functions (linear, quadratic, degree, trigonometric), and to establish metric correlations between different functions. It is possible to use the properties of monotone functions [20, p. 35]: "*if the function* f *increases on a segment* $[a;b]$*, then* $min_{x\in[a;b]}f(x) = f(a)$*,* $max_{x\in[a;b]}f(x) = f(b)$*; if the function* f *decreases on a segment* $[a;b]$*, then* $min_{x\in[a;b]}f(x) = f(b)$*,* $max_{x\in[a;b]}f(x) = f(a)$". These two properties can be combined into one and be more broadly formulated: "*monotonic function on the segment takes its smallest and largest values at the ends of that segment*". This formulation may make it somewhat easier to find the extremes of a function on a segment.

Here is an example of finding the distance between two linear functions in the space $C_{[a;b]}$.

Example 10. Consider two points y_1 and y_2 of space $C_{[a;b]}$, which are linear functions: $y_1 = k_1x + b_1$ and $y_2 = k_2x + b_2$. Let's find the distance between these points by the space metric $C_{[a;b]}$. In Example 7, we have already considered the particular case of linear functions: $y = kx$. Due to the absence of a free term on the right side of the equality, find the distance between such functions was easy enough. In our case, consider the function:

$$y = |y_1 - y_2| = |(k_1x + b_1) - (k_2x + b_2)| = |(k_1 - k_2)x + (b_1 - b_2)|.$$

May be the following cases of the plot the graphs of the functions $y = k_1x + b_1$ and $y = k_2x + b_2$. Let's the equalities $k_1 = k_2 = 0$ be satisfied. In this case, the graphs of both functions are parallel to the OX axis. Then by the equality (8) we have:

$$\rho(y_1; y_2) = max_{x\in[a;b]}|y_1 - y_2| = max_{x\in[a;b]}|(k_1 - k_2)x + (b_1 - b_2)| =$$
$$= max_{x\in[a;b]}|(k_1 - k_2)x + (b_1 - b_2)| = max_{x\in[a;b]}|b_1 - b_2| = |b_1 - b_2|.$$

We consider the different points of space, so $b_1 \neq b_2$.

Now let's at least one of the numbers k_1 or k_2 is non-zero, for example, $k_1 \neq 0$. Suppose that the graphs of the functions y_1 and y_2 on the segment $[a; b]$ do not intersect. In this case, one of them is higher than the other at each point in the segment. For example, if the inequality $y_1 > y_2$ at every point of the segment $[a; b]$ is executed then we have:

$$y = |y_1 - y_2| = y_1 - y_2 = (k_1 - k_2)x + (b_1 - b_2).$$

Since the function $y = (k_1 - k_2)x + (b_1 - b_2)$ is linear and therefore monotonous, it takes its smallest and largest values at the ends of the segment. Therefore, we will have:

$$\rho(y_1; y_2) = max_{x\in[a;b]}|y_1 - y_2| = max_{x\in[a;b]}\big((k_1 - k_2)x + (b_1 - b_2)\big) =$$
$$= max\,\{(k_1 - k_2)a + (b_1 - b_2);\ (k_1 - k_2)b + (b_1 - b_2)\}.$$

This maximum can be reached not only at the end of the segment. In particular, if case of equality $k_1 = k_2$, the maximum is reached at each point of the segment $[a; b]$, in this case the graphs of the functions are parallel to each other.

Now consider the case when the graphs of both functions are intersected. It will mean that the function $y = (k_1 - k_2)x + (b_1 - b_2)$ intersects the OX axis (or touches it) at the point x_0 of the segment $[a; b]$, where

$$x_0 = \frac{b_2 - b_1}{k_1 - k_2}.$$

In this case, the function

$$y = |(k_1 - k_2)x + (b_1 - b_2)|$$

will take non-negative values on the throughout segment $[a; b]$, this function will be monotone on each of the segments $[a; x_0]$ and $[x_0; b]$, and will be value zero at the point x_0. Therefore, this function can take the largest value only at the ends of the segment $[a; b]$.

Since we have considered all possible cases of the mutual placement of both functions, we can conclude that the distance between two linear functions by the metric of space $C_{[a;b]}$ is equal to the greater of the absolute values of the differences of the values of both functions at the ends of the segment $[a; b]$.

To generalize of Example 10 in the case of arbitrary two monotonic and even continuous functions is impossible.

According to the results Examples 6-8 may suggest that a rectilinear placement of points of a metric space requires a certain "monotony" of the placement of these points. However, this is not always true.

Example 11. On the segment $[0; 1]$ we consider the functions:

$$y_1 = x, y_2 = -x, y_3 = -x + 1, y_4 = x - 1.$$

We show that in space $C_{[a;b]}$ these functions are rectilinear placed. For this we find by the space metric $C_{[a;b]}$ the distances between these functions (points). By the formula (8) we have:

$$\rho_{12} = 2, \rho_{13} = 1, \rho_{14} = 1, \rho_{23} = 1, \rho_{24} = 1, \rho_{34} = 2.$$

Since the equality is executed: $\rho_{12} = 2 = 1 + 1 = \rho_{13} + \rho_{23}$, then the points y_1, y_2, y_3 are rectilinear placed in the space $C_{[a;b]}$, and the point y_3 lies between the points y_1 and y_2, that is, the point y_3 is internal for points y_1, y_2, y_3. Similarly, from the equality $\rho_{12} = 2 = 1 + 1 = \rho_{14} + \rho_{24}$ it follows that the points y_1, y_2, y_4 are also rectilinear placed and the point y_4 is internal to them.

On the other hand, the equality $\rho_{34} = 2 = 1 + 1 = \rho_{13} + \rho_{14}$ indicates that the points y_1, y_3, y_4 are rectilinear placed and the point y_1 lies between the points y_3 and y_4. In addition, from the equality $\rho_{34} = 2 = 1 + 1 = \rho_{23} + \rho_{24}$ we get the points y_2, y_3, y_4 are also rectilinear placed and the point y_2 is internal to them.

Since we have consider all possible triples of points and all of them are rectilinear placement, then, by Definition 4, all four points are rectilinear placement in space $C_{[a;b]}$.

Note that each of the four points lies between some two of them, that is, there are no extreme points between these points. This situation cannot be in Euclid's geometry. There, from the four points lying on a straight line, two will be extreme and two will be internal to those points. Therefore, in this example, as in Examples 5 and 9, we are dealing with elements of non-Euclidean geometry.

The peculiarity of placement of points y_1, y_2, y_3, y_4 can be explained by moving away from the intuitive perception of straightness. It can be illustrated by the example of the space of points of a single circle. If the distance between the two points of the circle is the length of the smaller of the two arcs of the circle connecting these points, it is easy to make sure that the space becomes metric. In this case, the points y_1, y_2, y_3, y_4 will be the ends of two mutually perpendicular diameters of the circle.

The concept of rectilinear in metric space is ambiguous. In Euclid geometry, two points define the only direct line containing these points; this fact is established by the corresponding axiom [27, p. 3]. In metric space, the situation is different. Let's demonstrate by the example of linear functions the ambiguity of the notion of a rectilinear placement of points in space $C_{[a;b]}$.

Example 12. Consider the functions: $y_1 = x + 1$, $y_2 = x$, $y_3 = x - 2$, $y_4 = -x$, given on the segment $[0; 1]$.

Find the distances between these functions by the formula (8), taking into account the results of Example 10:

$$\rho_{12} = 1; \rho_{13} = 3; \rho_{14} = 3; \rho_{23} = 2; \rho_{24} = 2; \rho_{34} = 2.$$

From the obtained equalities, it follows that the points (functions) y_1, y_2, y_3 are rectilinear placed, since equality is executed: $\rho_{13} = 3 = 1 + 2 = \rho_{12} + \rho_{23}$. In this case, the point y_2 lies between the points y_1, y_3.

On the other hand, the points y_1, y_2, y_4 are also rectilinear placed because the equality is executed: $\rho_{14} = 3 = 1 + 2 = \rho_{12} + \rho_{24}$. In this case, the point y_2 also lies between the points y_1, y_4.

Since in both cases the points y_1, y_2 are present in each of the equalities, then in Euclid's geometry, all four points y_1, y_2, y_3, y_4 must belong to one straight line. However, from the obtained values of the distances between them, it follows that the points y_2, y_3, y_4 cannot belong to one straight line, since distances between them are equal (they form an equilateral triangle whose lengths are 2). At the same time, the graphs of the functions are arranged one below the other, that is, at each point of the segment $[0; 1]$, the following inequalities are executed:

$$y_1 \geq y_2 \geq y_4 \geq y_3. \tag{9}$$

This fact indicates that in the space $C_{[a;b]}$ the concept of rectilinear placement differs in its properties from the concept of straightness in Euclid's geometry. Below we show that when you change the space metric, the property of placement monotony can be preserved.

The results obtained can be explained by the fact that the properties of the rectilinear placement of points in Euclid geometry are fixed by the corresponding postulates (axioms) [27, pp. 3-5]. In the metric space there are only axioms of the distance between the points, and therefore a certain ambiguity of the notion of straightness is possible.

Assuming a slightly different interpretation of rectilinear placement, then the obtained results are easy to understand. For example, moving around the globe as straightness, we however moving in a circle centered at the center of the Earth. Therefore, by imposing certain restrictions on the path that you can move from point to point, and for the distance between points by taking the shortest path length between points, you can get a non-Euclidean interpretation of the rectilinear placement of points.

Consider another example that demonstrates another feature of the space $C_{[a;b]}$. If you place three points on a straight line in Euclid's geometry, and move one of the two end points along that line in the direction of the other two points, then moving continuously, this point first coincides with one of the other two points (internal point), and then with the second (other outside) point, after which it will continue to move along this line. In the space $C_{[a;b]}$ this is not always possible to do.

Example 13. Consider the functions: $y_1 = 0$, $y_2 = x$, $y_3 = C$, where C is a constant. In Example 6, we have shown that all points of the space R^1 are rectilinear placement, so by changing the constant C we move this point in a straight line, both in this space and in the space $C_{[a;b]}$ (see Example 10).

First, consider the case where the constant C satisfies the inequality: $0 < C \leq 0{,}5$. In this case, we will found the distance between the points (functions) y_1, y_2, y_3 by the formula (8), we have:

$$\rho_{12} = 1;\ \rho_{13} = C = |C|;\ \rho_{23} = 1 - C = 1 - |C|.$$

Since the equality is executed:

$$\rho_{12} = 1 = |C| + (1 - |C|\,) = \rho_{13} + \rho_{23},$$

then the points y_1, y_2, y_3 are rectilinear placement in the space $C_{[a;b]}$.

At $C < 0$ the distances between points (functions) y_1, y_2, y_3 will be:

$$\rho_{12} = 1;\ \rho_{13} = -C = |C|;\ \rho_{23} = 1 - C = 1 + |C|.$$

Since the equality is executed:

$$\rho_{23} = 1 + |C| = \rho_{12} + \rho_{13},$$

then points y_1, y_2, y_3 are rectilinear placement in the space $C_{[a;b]}$.

Now let the constant C satisfy the inequality: $C > 0{,}5$. In this case, the distances between the points (functions) y_1, y_2, y_3 will be: $\rho_{12} = 1$; $\rho_{13} = \rho_{23} = C$. With these values of distances, the points y_1, y_2, y_3 form an isosceles triangle with a length of base which equal to ρ_{12} and with sides whose lengths are equal to C (constant C exceeds 0.5). Thus, for the points y_1, y_2, y_3, the rectilinear placement in the space $C_{[a;b]}$ was broken, although the point y_3 moved straightforwardly.

Example 13 indicates that, despite the equality of all points of the metric space, in specific spaces not only the metric, but also the intrinsic properties of each point (element) of the space can significantly affect the geometric properties of the entire space.

After studying the definite integral and its applications, it is possible to acquaint the pupils with another metric space related to the geometric content of the definite integral. These topics are studied in the eleventh grade [38, pp. 254-256; 39, pp. 373-374; 40, pp. 235-238; 41, pp. 112-113].

Consider the set of functions continuous on the segment $[a; b]$. For the distance between the two functions $f(x)$ and $g(x)$ of this set we take the number which is given by the formula:

$$\rho(f; g) = \int_a^b |f(x) - g(x)| dx \quad (10)$$

With this choice of metric, the considered set of functions becomes a metric space denoted by C_L.

To test the distance axiom in this space, it is necessary to know a number of properties of a defined integral that are not actually mentioned in current textbooks. In particular, it is not mentioned the monotonicity property of a defined integral:

"*If the functions $f(x)$ and $g(x)$ are continuous on the segment $[a;b]$, and inequality $f(x) \geq g(x)$ is executed at every point in this segment, then the inequality*

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

is executed."

This property is quite simple to obtain the defined integral is determined as the boundary of the integral sum [39, pp. 367-368; 40, pp. 225-226; 41, pp. 112-113]. Such a definition is historically justified and makes it quite easy to obtain various applications of a definite integral. For calculating defined integrals of functions is more convenient to use the Newton-Leibniz formula.

It should be noted that, as a rule, only the arithmetic actions over integrals are mentioned among the properties of a defined integral in the existing mathematical textbooks, and the condition of the existence of the integral is not actually paid attention. Therefore, pupils should be emphasized that function which is continuous on the segment is integrated, that is, continuity of a function on a segment is a sufficient condition for exist its integral on that segment. In our view, it will greatly enhance the importance of studying the concept of continuity of function. The condition of continuity of the integrand function can be included when familiarizing with the Newton-Leibniz formula [39, p. 360; 40, p. 223].

Let's make verification of distance axioms for the metric given by formula (10). From continuity of the functions $f(x)$ and $g(x)$ follows continuity, the function $|f(x) - g(x)|$ is existed, so the distance defined by formula (10) exists. Any integral sum for the function $|f(x) - g(x)|$ on the interval $[a;b]$ is the sum of the nonnegative terms, and therefore its boundary cannot be negative. The triangle inequality for the functions $f(x)$, $g(x)$, $h(x)$, continuous on the segment $[a;b]$, is obtained using the monotonicity property of a defined integral:

$$\rho(f;g) = \int_a^b |f(x) - g(x)|dx = \int_a^b \left|\big(f(x) - h(x)\big) + \big(h(x) - g(x)\big)\right|dx \leq$$
$$\leq \int_a^b (|f(x) - h(x)| + |h(x) - g(x)|)dx = \int_a^b |f(x) - h(x)|dx + \int_a^b |h(x) - g(x)|dx =$$
$$= \rho(f;h) + \rho(h;g).$$

Geometrically, the distance between the two functions $f(x)$ and $g(x)$ of the space C_L defines the figure area which is bounding by the graphs of these functions on the segment $[a;b]$.

In school textbooks the less general formula of the area of a figure bounded by the graphs of the functions $f(x)$ and $g(x)$, which satisfy the inequality $f(x) \geq g(x)$ on the segment $[a;b]$ is used:

$$S = \int_a^b (f(x) - g(x)\,dx$$

[38, p. 261; 39, pp. 373-374; 40, pp. 236-237; 41, p. 115].

The metric defined by formula (10) is less sensitive to the features of the structure of an individual element of space. It can be illustrated by the following example.

Example 14. Let us return to Example 12 and consider the functions: $y_1 = x + 1$, $y_2 = x$, $y_3 = x - 2$, $y_4 = -x$, given on the segment $[0;1]$. Previously, we found that all four functions are not rectilinear placement in $C_{[a;b]}$. Let's find the distances between them by the metric of space C_L. By the formula (10) we have:

$$\rho_{12} = \int_0^1 |(x+1) - x|dx = \int_0^1 dx = 1 - 0 = 1\,;$$

$$\rho_{13} = \int_0^1 |(x+1)-(x-2)|dx = \int_0^1 3dx = 3(1-0) = 3;$$

$$\rho_{14} = \int_0^1 |(x+1)-(-x)|dx = \int_0^1 (2x+1)dx = (1^2-0^2)+(1-0) = 2;$$

$$\rho_{23} = \int_0^1 |x-(x-2)|dx = \int_0^1 2dx = 2(1-0) = 2\,;$$

$$\rho_{24} = \int_0^1 |x-(-x)|dx = \int_0^1 2xdx = 1^2-0^2 = 1\,;$$

$$\rho_{34} = \int_0^1 |(x-2)-(-x)|dx = \int_0^1 (2-2x)dx = 2(1-0)-(1^2-0^2) = 1.$$

Since the equality is executed: $\rho_{13} = 3 = 1+2 = \rho_{12}+\rho_{23}$, then the points y_1, y_2, y_3 are rectilinear placed in the space C_L, and the point y_2 lies between the points y_1 and y_3.

From the equality $\rho_{14} = 2 = 1+1 = \rho_{12}+\rho_{24}$ it follows that the points y_1, y_2, y_4 are also rectilinear placement, and the point y_2 lies between the points y_1 and y_4.

Since the equality is executed : $\rho_{13} = 3 = 2+1 = \rho_{14}+\rho_{34}$, then the points y_1, y_3, y_4 are rectilinear placement in the space C_L, and the point y_4 lies between the points y_1 and y_3.

From the equality $\rho_{23} = 2 = 1+1 = \rho_{24}+\rho_{34}$ it follows that the points y_2, y_3, y_4 are also rectilinear placement, and the point y_4 lies between the points y_2 and y_3.

We considered all four possible triples of points, and each of them turned out to be rectilinear placement. By Definition 4, all four points are rectilinear placement in C_L, and the points y_1, y_3 are external, and the points y_2, y_4 are internal to the points y_1, y_2, y_3, y_4. All points are placed in the following order: y_1, y_2, y_4, y_3. This placement of functions coincides with the order of their placement on the coordinate plane.

Example 14 is a consequence of the more general property of C_L space. Comparing the results of Examples 12 and 14, we can conclude that the metric of the space C_L is more "stronger" than the metric of the space $C_{[a;b]}$, since even the "monotonicity" of the placement of functions could not ensure their rectilinear placement in the space $C_{[a;b]}$. In the space C_L, unlike the space $C_{[a;b]}$, the rectilinear placement of points can provide them with a certain "monotonicity of placement", as evidenced by the following example.

Example 15. Consider the set F of functions continuous on a segment $[a;b]$. Let's for any function $f(x)$ and any function $g(x)$ of this set, the inequality $f(x) \geq g(x)$ is executed at every point of this segment.

We show that the set F is rectilinear placement in the space C_L. To do this, we consider arbitrary three elements $f(x)$, $g(x)$, $h(x)$ of this set. Let for them, at each point x of the segment $[a;b]$ are satisfied, for example, inequalities $f(x) \geq g(x) \geq h(x)$. By formula (10) we will find the distances between these elements:

$$\rho(f;g) = \int_a^b |f(x)-g(x)|dx = \int_a^b \big(f(x)-g(x)\big)dx = \int_a^b f(x)dx - \int_a^b g(x)dx\,;$$

$$\rho(f;h) = \int_a^b |f(x)-h(x)|dx = \int_a^b \big(f(x)-h(x)\big)dx = \int_a^b f(x)dx - \int_a^b h(x)dx\,;$$

$$\rho(g;h) = \int_a^b |g(x)-h(x)|dx = \int_a^b \big(g(x)-h(x)\big)dx = \int_a^b g(x)dx - \int_a^b h(x)dx.$$

From the obtained equalities follows the equality:

$$\rho(f;h) = \int_a^b f(x)dx - \int_a^b h(x)dx =$$

$$= \left(\int_a^b f(x)dx - \int_a^b g(x)dx \right) + + \left(\int_a^b g(x)dx - \int_a^b h(x)dx \right) = \rho(f; g) + \rho(g; h).$$

The obtained equality means that the elements $f(x)$, $g(x)$, $h(x)$ of the set F are rectilinear placement in the space C_L.

Since we have chosen elements arbitrarily, by Definition 4, the whole set F is rectilinear placement in the space C_L. Now, the result of Example 14, due to inequality (9), becomes a special case of Example 15.

4. Conclusions

The current state of development of non-Euclidean geometries, and the level of their application, indicates the need to introduce into the educational process of the basic school of the basic concepts and elements of these geometries. This paper presents examples of such implementation in geometry lessons, as well as in extracurricular work in mathematics. These examples are based on the school mathematics course, and form generalized notions of point, distance between points, and rectilinear placement of points. It is proposed to introduce these concepts throughout the course of school mathematics, starting from the 7$^{\text{th}}$ grade.

Formation of generalized notions of distance between points, and straightness of their location, can be started in the seventh grade with advanced study of mathematics. At the same time, examples of ambiguity of these concepts should be used to make understandable for students of the appropriate age.

Systematic introduction to non-Euclidean geometry students can begin in the ninth grade with advanced study of mathematics. Such elements can be demonstrated when studying the numerical straight and coordinate planes.

In senior classes with advanced study of mathematics, at the profile level, it is possible to introduce the notion of metric space, and as examples of such spaces we can consider the spaces of continuous and integrated functions on the segment. However, in our opinion, this material should be considered in elective classes in mathematics.

The material presented in this paper can be considered as the first acquaintance with the basics of metric geometry. Following these examples, you can develop and solve a large number of different problems about the mutual placement of basic elementary functions in different metric spaces, and build, explore different geometric forms in these spaces. In [42, 43] the question of rectilinear placement of points of metric space was applied to the study of geometric properties of this space.

Further research will focus on the application of the generalized concept of angle formed by the points of the metric space, and the concept of flat placement of points of this space. The research is planned to be limited to discrete cases of rectilinear and flat placement of points.

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