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Про фрактальні властивості деяких множин канторівського типу, пов'язаних з Q-зображенням дійсних чисел

Наводяться результати дослідження тополого-метричних і фрактальних властивостей множин дійсних чисел з обмеженнями на вживання символів в їх Q-зображенні (узагальнення s-адичних розкладів)

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On fractal properties of some Cantor-like sets related to Q-representation of real numbers

In the paper, we study topological, metric and fractal properties of the sets of numbers with conditions on use of digits in their Q-representation (generalization of s-adic expansions)

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1 Introduction

Definition 1.1. Let $\triangle^3_{c_1\ldots c_k\ldots}$ be a formal (symbolic) representation of number $x \in [0,1]$ in ternary numeration system, $c_i = c_i(x) \in \{0,1,2\}$, i.e.,

$$x \equiv \triangle^3_{c_1\ldots c_k\ldots} = \sum_{m=1}^{\infty} 3^{-m} c_m.$$

Let $N_i(x,n) = \#\{k : c_k(x) = i, k \leqslant n\}$ be a number of digits "i" in ternary expansion of number x to n-th position inclusive, $i = 0,1,2$. If limit $\lim\limits_{n\to\infty} n^{-1} N_i(x,n) = \nu_i(x)$ exists, then value $\nu_i(x)$ is called the *frequency (or asymptotic frequency) of digit* "i" in ternary representation of x.

In paper [?], the continuum set of fixed points of mapping $y = \nu_1^3(x)$ where $\nu_1^3(x)$ is a function of frequency of digit 1 in ternary representation of x is described. The points of this set have the following properties:

1. Every fifth ternary digit can be chosen arbitrarily.

2. Other digits are obtained by algorithm and depend on all previous ternary digits of x.

The Hausdorff-Besicovitch dimension of this set is found in paper [?].

This motivates our interest in the problem about fractal properties of the set $M \subset [0,1]$ consisting of the numbers such that their Q-representations (generalization of s-adic expansion) have similar structural properties. Namely, we study the set M with the following properties:

1. Every l-th ($1 < l \in N$) Q-symbol of $x \in M$ is arbitrary.

2. Q-symbol with number $n \notin \{1 + kl\}$, $k = 0,1,2,\ldots$ is determined uniquely and depend on all previous Q-symbols.

Does dependence of n-th Q-symbol of $x \in M$ on previous Q-symbols influence the Hausdorff-Besicovitch dimension of the set M? It is easy to prove that it does not influence if Q-representation is at least an s-adic representation. Our paper is devoted to these and some other problems.

2 s-symbol Q-representation of real number

Let s be a fixed positive integer, $s > 1$, let $A = \{0,1,\ldots,s-1\}$, and let $Q = \{q_0, q_1, \ldots, q_{s-1}\}$ be a fixed set with the following properties:

$$\begin{cases} 1)\ \ q_i > 0; \\ 2)\ \ q_0 + q_1 + \ldots + q_{s-1} = 1 \end{cases}, \qquad (1)$$

$\beta_0 = 0$, $\beta_j = q_0 + q_1 + \ldots + q_{j-1}$.

Theorem 2.1 ([?, p. 87]). *For any number $x \in [0,1]$ there exists a sequence of numbers $\alpha_k \in A$ such that*

$$x = \beta_{\alpha_1} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k} \prod_{j=1}^{k-1} q_{\alpha_j} \right]. \qquad (2)$$

For any real number u there exists an expansion

$$u = [u] + \beta_{\alpha_1(u)} + \sum_{k=2}^{\infty} \left[\beta_{\alpha_k(u)} \prod_{j=1}^{k-1} q_{\alpha_j(u)} \right] \qquad (3)$$

where $[u]$ *is a floor function of* u.

Definition 2.1. Representation of the number x (number u) by the series (**??**) (series (**??**)) is called the *s-symbol Q-expansion.*

We denote symbolically expression (**??**) by $\triangle^Q_{\alpha_1\ldots\alpha_k\ldots}$ and call it by *s–symbol Q-representation* of x. The number $\alpha_k(x)$ is called the k-th Q-symbol of x.

Remark 1. *If* $q_i = s^{-1}$, $i = \overline{0, s-1}$, *then Q-expansion is an s-adic expansion and Q-representation is a representation of number in numeration system with base s.*

Theorem 2.2 ([**?**])**.** *Any number* $x \in [0,1]$ *have no more than two formally different Q-representations. There exist numbers with two different Q-representations, one has period* (0)*, other has period* $(s-1)$.

Definition 2.2. Numbers having period 0 in their Q-representations are called the *Q-rational*, the rest are the *Q-irrational.*

Any Q-rational number has two different Q-representations, and any Q-irrational number has a unique Q-representation.

The set of all Q-rational numbers is a countable set.

Remark 2. *The notion of Q-symbol is well defined for Q-irrational number and is not well defined for Q-rational number. Therefore, we shall give more information in the sequel to avoid ambiguity.*

3 Function of frequency of digits of Q-representation

Let $N_i(x,k)$ be a number of symbols "i" in Q-representation of number x to k-th position inclusive. Then limit (if it exists)

$$\lim_{k\to\infty} k^{-1} N_i(x,k) = \nu_i(x)$$

is called the *frequency* of symbol "i" in Q-representation of x.

It is evident that the frequency of Q-symbol does not depend on arbitrary finite amount of symbols of this number.

For Q-rational numbers, the frequency of symbol 0 (or $s-1$) is equal to 1, and the frequencies of the rest symbols are equal to 0.

For any fixed i, function $f(x) = \nu_i(x) = u$ is an everywhere discontinuous function. Moreover, this function takes any value from $[0,1]$ on the continuum set. Furthermore,

$$E_u = \{x : \nu_i(x) = u\}$$

is a dense set in $[0,1]$, and its Hausdorff-Besicovitch dimension is equal to

$$\alpha_0(E_u) = \sup_{(\tau_0\tau_1\ldots\tau_{s-1})} \frac{\ln \tau_0^{\tau_0}\tau_1^{\tau_1}\ldots\tau_{s-1}^{\tau_{s-1}}}{\ln q_0^{\tau_0}q_1^{\tau_1}\ldots q_{s-1}^{\tau_{s-1}}}$$

where $\tau_k > 0$, $\tau_i = u$, $\tau_0 + \tau_1 + \ldots + \tau_{s-1} = 1$.

The property W of elements of the set M is called a *normal property* if almost all elements of M have this property. There exist a few mathematical notions allowing to interpret uniquely words "almost all". The notions of cardinality, measure, Hausdorff-Besicovitch dimension, Baire category are among them. We use the notion of measure of set (Lebesgue measure).

Definition 3.1. The number $x = \triangle^Q_{\alpha_1\ldots\alpha_k\ldots}$ such that the frequency $\nu_i(x)$ of Q-symbol i satisfies the condition

$$\nu_i(x) = q_i \qquad \forall i \in \{0, 1, \ldots, s-1\}$$

is called a *Q-normal number.*

From the following proposition it follows that this definition is well defined.

Theorem 3.1. *The Lebesgue measure of all Q-normal numbers from* $[0,1]$ *is equal to 1.*

Theorem 3.2. *Almost all numbers from* $[0,1]$ *are normal in any Q-representation with rational* $q_0, q_1, \ldots, q_{s-1}$.

Definition 3.2. The number $x \in [0,1]$ is called a *non-normal* in Q-representation if x does not have frequency for at least one Q-symbol.

Theorem 3.3. *The set* $V \subset [0,1]$ *of non-normal in Q-representation numbers is a superfractal, i.e., it is a continuum set, and its Hausdorff-Besicovitch dimension is equal to 1.*

4 Transition from s-symbol Q-representation to adjusted s^l-symbol $\overline{Q}$-representation

Let l be a fixed positive integer, $l > 1$, and let $(a_1, a_2, \dots, a_l) \in A^l$.

Define a simple function

$$k = \varphi(\alpha_1, \alpha_2, \dots, \alpha_l) = \alpha_1 s^{l-1} + \alpha_2 s^{l-2} + \dots + \alpha_l$$

and put $\overline{q}_k = \prod_{j=1}^{l} q_{\alpha_j}$.

Lemma 1. *If the set $Q = \{q_0, \dots, q_{s-1}\}$ satisfies conditions (??), then the set $\overline{Q} = \{\overline{q_0}, \dots, \overline{q_m}\}$, $m = s^l - 1$, also satisfies conditions (??).*

Proof. Let $(\alpha_1 \dots \alpha_l)_s$ be an s-adic representation of number $k \in N \cup \{0\}$, i.e.,

$$k = \alpha_1 s^{l-1} + \alpha_2 s^{l-2} + \dots + \alpha_l = (\alpha_1 \dots \alpha_l)_s.$$

Let $\alpha_1 \dots \alpha_l = \overline{k}$, namely,

$$\underbrace{0 \dots 00}_{l} = \overline{0},$$

$$\underbrace{0 \dots 01}_{l} = \overline{1},$$

$$\dots\dots\dots\dots$$

$$\underbrace{(s-1) \dots (s-1)}_{l} = \overline{m},$$

$$((s-1) \dots (s-1)(s-1))_s =$$
$$\frac{s-1}{1-s}\left(1 - s^l\right) = s^l - 1 = m.$$

Divide all infinite sequence of Q-symbols of x into blocks consisting of l symbols. Then Q-representation of x can be rewritten formally by infinite ordered set of symbols from the set $\{\overline{0}, \overline{1}, \dots, \overline{m}\}$. Namely,

$$x = \Delta^{Q}_{\alpha_1 \dots \alpha_k \dots} = \Delta^{\overline{Q}}_{k_1 \dots k_n \dots}$$

where $k_1 \equiv (\alpha_1 \dots \alpha_l)_s$, $\dots$,
$k_{n+1} \equiv (\alpha_{1+nl+1} \dots \alpha_{l+nl+l})_s$,
$k_1 = \alpha_1 s^{l-1} + \alpha_2 s^{l-2} + \dots + \alpha_l$.
$\overline{Q} = \{\overline{q}_0, \overline{q}_1, \dots, \overline{q}_m\}$, where
$k_1 = \alpha_1 s^{l-1} + \alpha_2 s^{l-2} + \dots + \alpha_l$.
$\overline{q}_k = \prod_{j=1}^{l} q_{\alpha_j}$.

It is evident that $\begin{cases} \overline{q}_k > 0, \\ \sum_{k=0}^{m} \overline{q}_k = 1. \end{cases}$ □

Remark 3. *If we have the set Q satisfying (??), by algorithm from Lemma 1, we can construct the new set $\overline{Q}$ satisfying conditions (??). This set defines new representation adjusted with Q-representation of $x \in [0, 1]$.*

Let

$$\varphi(x_1, x_2, \dots, x_l) = x_1 s^{l-1} + x_2 s^{l-2} + \dots + x_{l-1} s + x_l.$$

Theorem 4.1. *For any $x \in [0, 1]$ the equality holds:*

$$x \equiv \Delta^{Q}_{\alpha_1 \dots \alpha_n \dots} = \Delta^{\overline{Q}}_{k_1 \dots k_n \dots} \quad (4)$$

where $k_i = \varphi(\alpha_{l(i-1)+1}, \alpha_{l(i-1)+2}, \dots, \alpha_{li})$ for any $i = 1, 2, \dots$.

Proof. In fact, α_i takes values from the set A. Then functional φ of l variables takes s^l different values from the set $\overline{A}$,

$$\overline{A} = \big\{0 = \varphi(0, \dots, 0), 1 = \varphi(0, \dots, 0, 1), \dots,$$
$$s^l - 1 = \varphi(s-1, \dots, s-1)\big\}.$$

To conclude the proof it is enough to show the equality of cylinders

$$\triangle^{Q}_{\alpha_1 \dots \alpha_{lm}} = \triangle^{\overline{Q}}_{k_1 \dots k_m}$$

for any $m \in N$ and sequence $(\alpha_1, \dots, \alpha_{lm})$ where k_i are given by formulae (??).

Using Lemma 1, we get

$$|\triangle^{\overline{Q}}_{k_1 \dots k_m}| = \prod_{j=1}^{m} \overline{q}_{k_j} = \prod_{i=1}^{lm} q_{\alpha_i} = |\triangle^{Q}_{\alpha_1 \dots \alpha_{lm}}|.$$

It remains to prove that

$$\inf \triangle^{\overline{Q}}_{k_1 \dots k_m} = \inf \triangle^{Q}_{\alpha_1 \dots \alpha_{lm}}.$$

Indeed,

$$\inf \triangle^{\overline{Q}}_{k_1 \dots k_m} = \triangle^{\overline{Q}}_{k_1 \dots k_m (0)} =$$

$$= \overline{\beta}_{k_1} + \sum_{n=2}^{m} \left[\overline{\beta}_{k_n} \prod_{j=1}^{n-1} \overline{q}_{k_j} \right] =$$

$$= \beta_{\alpha_1} + \sum_{n=2}^{lm} \left[\beta_{\alpha_n} \prod_{j=1}^{n-1} q_{\alpha_j} \right] = \inf \triangle^{Q}_{\alpha_1 \dots \alpha_{lm}},$$

which proves the theorem. □

5 Fractal properties of some Cantor-like sets related to Q-representation of real numbers

***1. The set** M.*

Consider two positive integers $s > 1$, $l > 1$ and a sequence of matrices

$$\|\mathbf{c_{ij}^n}\| = \begin{pmatrix} c_{01}^n & c_{02}^n & \dots & c_{0(l-1)}^n \\ c_{11}^n & c_{12}^n & \dots & c_{1(l-1)}^n \\ \dots & \dots & \dots & \dots \\ c_{(s-1)1}^n & c_{(s-1)2}^n & \dots & c_{(s-1)(l-1)}^n \end{pmatrix}$$

where $n = 1, 2, \dots$, $c_{ij}^n \in A = \{0, 1, \dots, s-1\}$.

Theorem 5.1. *If the sequence* $\left(||c_{ij}^n||\right)_{n=1}^{\infty}$ *is a purely periodic with period*

$$\left(||c_{ij}^{n_1+1}||, \dots, ||c_{ij}^{n_1+2}||, ||c_{ij}^{n_1+p}||\right),$$

then Hausdorff-Besicovitch dimension of the set

$$M = \{x : x = \triangle^Q_{\alpha_1 \dots \alpha_k \dots}, \alpha_{1+(n-1)l}(x) \in A, \\ \alpha_{1+(n-1)l+j} = c^n_{\alpha_{1+(n-1)l}j}, j = \overline{1, l-1}\}$$

is a root of the equation

$$\sum_{i_1=0}^{s-1} \dots \sum_{i_p=0}^{s-1} \left(\prod_{k=1}^{p} \left[q_{i_k} \prod_{j=1}^{l-1} q_{c^k_{i_k j}} \right] \right)^x = 1$$

or

$$\prod_{k=1}^{p} \sum_{i=0}^{s-1} q_i^x \prod_{j=1}^{l-1} q^x_{c^k_{ij}} = 1. \qquad (5)$$

Proof. M is a self-similar set, since for any sequence $(i_1, c^1_{i_1 0}, \dots, c^1_{i_1 l}, \dots, i_p c^p_{i_p 0}, \dots, c^p_{i_p l})$ and corresponding cylinder $\triangle_{i_1 c^1_{i_1 0} \dots c^1_{i_1 l} \dots i_p c^p_{i_p 0} \dots c^p_{i_p l}}$ the set M is similar to the part of M belonging to this cylinder, namely,

$$M \overset{k}{\sim} \left[M \cap \triangle_{i_1 c^1_{i_1 0} \dots c^1_{i_1 l} \dots i_p c^p_{i_p 0} \dots c^p_{i_p l}} \right]$$

where similarity ratio is given by formula

$$k = \prod_{k=1}^{p} \left(q_{i_k} \prod_{j=1}^{l-1} q_{c^k_{ij}} \right).$$

Since M is a perfect set (i.e., a closed set without isolated points), self-similar dimension of M coincides with Hausdorff-Besicovitch dimension of M and is a root of equation (**??**). □

Remark 4. *If the sequence of matrices* $\left(||c_{ij}^n||\right)_{n=1}^{\infty}$ *is periodic but period starts from position* $n_1 + 1$, *then the theorem remains valid. Then* M *is not a self-similar set but is a finite union of self-similar sets with the same structure of similarity.*

Corollary 5.1. *If all matrices of sequence* $\left(||c_{ij}^n||\right)_{n=1}^{\infty}$ *are identical, i.e.,* $c_{ij}^n = c_{ij}$, *then the Hausdorff-Besicovitch dimension of* M *is a root of the equation*

$$\sum_{i=0}^{s-1} \left[q_i \prod_{j=1}^{l-1} q_{c_{ij}} \right]^x = 1.$$

Corollary 5.2. *If*

$$(c_{i1} \dots c_{i(l-1)}) = (c_{01} \dots c_{0(l-1)}), \quad i = \overline{0, s-1},$$

then the Hausdorff-Besicovitch dimension of M *is a root of the equation*

$$(q_0^x + \dots + q_{s-1}^x) \prod_{j=1}^{l-1} q^x_{c_{0j}} = 1.$$

Corollary 5.3. *If* $q_i = \frac{1}{s}$, $i = \overline{0, s-1}$, *then the Hausdorff-Besicovitch dimension of the set*

$$M = \{x : x = \triangle^s_{\alpha_1 \dots \alpha_k \dots}, \alpha_{1+(n-1)l}(x) \in A, \\ \alpha_{1+(n-1)l+j} = const_{1+(n-1)l+j}, j = \overline{1, l-1}\}$$

is equal to $\frac{1}{l}$.

***2. The set** O.*

Let (m_n) be an increasing sequence of positive integers such that $m_{n+1} - m_n \geqslant 2$, $(c_n, c'_n) \in A^2$, $n = 1, 2, \dots$.

Theorem 5.2. *The set*

$$O = \{x : x = \triangle^Q_{\alpha_1 \dots \alpha_k \dots}, \\ (\alpha_{m_n}(x), \alpha_{m_n+1}(x)) \neq (c_n, c'_{n+1})\}$$

is a nowhere dense perfect set of zero Lebesgue measure.

Proof. 1. For any subinterval (a, b) from $[0, 1]$ there exists cylinder $\nabla_{\alpha_1 \alpha_2 \dots \alpha_k}$ of some rank k such that it is contained in (a, b). Then for $m_n > k$,

$$\nabla_{\alpha_1 \alpha_2 \dots \alpha_k \dots \alpha_{m_n - 1} c_{m_n} c_{m_n+1}} \cap O = \varnothing.$$

Hence, the set O is a nowhere dense set by definition.

The set O is perfect according to theorem about structure of perfect sets of real numbers [?].

2. Suppose $F_0 = [0,1]$, F_k is a union of cylinders of rank k for Q-representation such that interior points contains points of the set O, and

$$\overline{F}_{k+1} = F_k \backslash F_{k+1}.$$

Then $\lambda(F_{k+1}) = \lambda(F_k) - \lambda(\overline{F}_{k+1})$, $O \subset F_{k+1} \subset F_k$ $\forall k \in N$ and $\lambda(O) \leqslant \lambda(F_{k+1}) \to \lambda(O) \quad (n \to \infty)$.

Since

$$\lambda(F_{k+1}) = \frac{\lambda(F_{k+1})}{\lambda(F_k)} \cdot \frac{\lambda(F_k)}{\lambda(F_{k-1})} \cdot \ldots \cdot \frac{\lambda(F_1)}{\lambda(F_0)},$$

it follows that

$$\lambda(O) = \lim_{n\to\infty} \lambda(F_{k+1}) = \lim_{n\to\infty} \prod_{k=1}^{m} \frac{\lambda(F_k)}{\lambda(F_{k-1})} =$$

$$= \prod_{k=1}^{\infty} \frac{\lambda(F_k)}{\lambda(F_{k-1})} = \prod_{k=1}^{\infty} \left[1 - \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}\right].$$

Since $\dfrac{\lambda(\overline{F}_{k+1})}{\lambda(F_k)} \geqslant q^2_{\min} > 0$, the series $\sum\limits_{k=1}^{\infty} \dfrac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}$ is divergent, and the last infinite product does so. Thus $\lambda(O) = 0$. □

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