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NORMAL FUNCTORS IN THE CATEGORY OF NON-ARCHIMEDEAN UNIFORM SPACES

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We consider functors in the category of non-Archimedean uniform spaces and uniformly continuous maps generated by some normal functors in the category of compact Hausdorff spaces. We also show that any natural transformation of normal functors generates a natural transformation of the induced functors in the category of non-Archimedean uniform spaces.

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Рассматриваются функторы в категории неархимедовых равномерных пространств и равномерно непрерывных отображений, порожденные некоторыми нормальными функторами в категории компактных хаусдорфовых пространств. Доказано, что естественное преобразование нормальных функторов порождает естественное преобразование индуцированных функторов в категории неархимедовых равномерных пространств.

1. Introduction. Various constructions of general topology are functorial. E. Shchepin [16] defined general properties of functors in the category **COMP** of compact Hausdorff spaces and continuous maps. He introduced the notion of normal functor in the category **COMP** which includes the power functors, G-symmetric powers, the hyperspace functor and the probability measure functor. A. Chigogidze [5] introduced the notion of a normal functor in the category **TYCH** of Tychonov spaces and defined a canonical extension of normal functors from the category **COMP** onto the category **TYCH**.

Functors in the category of uniform spaces and uniformly continuous maps were considered by many authors. In particular, the hyperspaces of uniform spaces were introduced by Bourbaki and later investigated in [20]; the uniform structures on the spaces of probability measures of uniform spaces were studied in [6], [14].

The aim of this paper is to consider a counterpart of the notion of normal functor in the category of non-Archimedean uniform spaces and uniformly continuous maps. Recall that a uniform structure $\mathcal{U}$ on a completely regular Hausdorff space is called *non-Archimedean* (see, e.g., [10, 18, 17]) if for each entourage $U \in \mathcal{U}$, we have $UU = U$.

In [10], it is proved that a completely regular Hausdorff space admits a non-Archimedean uniform structure if and only if it is zero-dimensional. Recall that a topological space is *zero-dimensional* if this space has a base consisting of open and closed sets.

In the present paper, we show that every normal functor in the category **COMP** determines a functor in the category of non-Archimedean uniform spaces and also that any natural

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transformation of normal functors in the category **COMP** determines a natural transformation of the corresponding functors in the category of non-Archimedean uniform spaces.

The results of this paper are tightly connected with the previous author's paper [15] in which he considered a construction that allows for defining normal functors in the category of ultrametric spaces and nonexpanding maps.

2. Preliminaries. Recall that a uniform structure on a set X is a family $\mathcal{U}$ of subsets of $X \times X$ satisfying the conditions:

1) for any $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$;
2) for any $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^{-1} \subset U$;
3) $\bigcap \mathcal{U} = \Delta_X = \{(x,x) \mid x \in X\}$ (the diagonal of $X \times X$).

A pair $(X,\mathcal{U})$, where $\mathcal{U}$ is a uniform structure on a set X, is called a *uniform space*.

Given a uniform structure $\mathcal{U}$, $U \in \mathcal{U}$, $x \in X$, we define $U(x) = \{y \in X \mid (x,y) \in U\}$ and similarly, for any $A \subset X$, $U(A) = \bigcup\{U(x) \mid x \in A\}$.

A family $\mathcal{A}$ is called U*-uniform*, where $U \in \mathcal{U}$, if, for every $A \in \mathcal{A}$, there exists $x \in X$ such that $A \subset U(x)$.

Let $(X_i,\mathcal{U}_i)$, $i = 1,2$, be uniform spaces. A map $f\colon X_1 \to X_2$ is called *uniformly continuous* if, for any $V \in \mathcal{U}_2$, there exists $U \in \mathcal{U}_1$ such that, for any $(x,y) \in U$, we have $(f(x_1), f(x_2)) \in V$. The uniform spaces and uniformly continuous maps form a category. We denote it by **UNIF**.

Let us recall the notion of complete uniform structure. A *Cauchy filter* in a uniform space X is a filter $\mathcal{F}$ such that, for every $U \in \mathcal{U}$, there exists $A \in \mathcal{F}$ such that $A \times A \subset U$. A uniform space is complete if every Cauchy filter in it converges.

A uniform structure $\mathcal{U}$ is called *non-Archimedean* if $U^2 \subset U$, for any $U \in \mathcal{U}$. In other words, a uniform structure is non-Archimedean if it consists of partitions. The non-Archimedean uniform spaces and their uniformly continuous maps form a category which we denote **NA**. A uniform space $(X,\mathcal{U})$ is *discrete* if $\Delta_X \in \mathcal{U}$.

Recall that a (pseudo)metric d on a set X is said to be an *ultra(pseudo)metric* if the following strong triangle inequality holds:

$$d(x,y) \le \max\{d(x,z), d(z,y)\} \text{ for all } x,y,z \in X.$$

It is easy to see that every ultrametric generates a non-Archimedean uniformity. On the other hand, every non-Archimedean uniform structure $\mathcal{U}$ on a set X is generated by a family of ultrapseudometrics $\{d_\alpha \mid \alpha \in A\}$ in the following sense: for any $U \in \mathcal{U}$, there exist $\alpha_1,\dots,\alpha_k \in A$ and $r > 0$ such that $\{(x,y) \in X \times X \mid d_{\alpha_i}(x,y) < r,\ i = 1,\dots,k\} \subset U$ and for every every $\alpha \in A$ there exist $r > 0$ and $V \in \mathcal{U}$ such that $V \subset \{(x,y) \in X \times X \mid d_\alpha(x,y) < r\}$.

2.1. Normal functors in the category of ultrametric spaces. Denote by **COMP** the category whose objects are compact Hausdorff spaces and whose morphisms are continuous maps. The notion of a normal functor in the category **COMP** is introduced by E.V. Shchepin [16].

In the sequel, 'functor' means 'covariant functor'.

Definition 1. We say that a functor $F\colon$ **COMP** $\to$ **COMP** is *normal* if:

1) F preserves weight (i.e., $w(F(X)) = w(X)$, for every infinite X);
2) F is continuous;
3) F is monomorphic (i.e., F preserves embeddings);
4) F is epimorphic (i.e., F preserves the onto maps);

5) F preserves intersections;
6) F preserves preimages;
7) F preserves singletons and the empty set.

The definition above requires some comments. Continuity of a functor F means that it commutes with the limits of inverse systems over directed sets.

For a monomorphic functor F and any closed subset A of a compact Hausdorff space X, we identify $F(A)$ with the subset $F(i)(F(A))$ of $F(X)$, where $i\colon A \to X$ denotes the inclusion map. That F preserves the intersections means that $F(\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} F(A_\alpha)$, for every family $\{A_\alpha \mid \alpha \in \Gamma\}$ of closed subsets of a compact Hausdorff space X. Given a monomorphic functor F that preserves the intersections, for any $a \in F(X)$, we define the *support* $\mathrm{supp}(a)$ as $\bigcap\{A \mid A$ is a closed subset of X and $a \in F(X)\}$. By $F_\omega(X)$ we denote the set of points of finite support in the set $F(X)$.

The preservation of preimages means that, for any map $f\colon X \to Y$ in **COMP** and any closed subset B of Y, we have

$$F(f)^{-1}(F(B)) = F(f^{-1}(F(B))).$$

2.2. Extension of normal functors onto the category of Tychonov spaces. By β we denote the Stone-Čech compactification functor acting from the category **COMP** to the category **TYCH** of Tychonov spaces and continuous maps.

The following construction is described by Chigogidze [5]. Given a normal functor $F\colon$ **COMP** $\to$ **COMP** and a Tychonov space X, we let

$$F_\beta(X) = \{a \in F(\beta X) \mid \mathrm{supp}(a) \subset X \subset \beta X\}.$$

If $f\colon X \to Y$ is a morphism in **TYCH**, then $F(\beta(f))(F_\beta(X)) \subset F_\beta(Y)$ and we denote by $F_\beta(f)$ the restriction $F(\beta(f))|F_\beta(X)\colon F_\beta(X) \subset F_\beta(Y)$. The obtained functor F_β in the category **TYCH** is a normal functor in the sense of [5].

For the sake of notational simplicity, we keep the notation F for the extended functor over the category **TYCH**.

3. Main results. Let F be a normal functor in the category **COMP**. We denote by F the corresponding functor in the category **TYCH**.

Let X be a Tychonov space such that there exists a non-Archimedean uniform structure $\mathcal{U}$ on X compatible with its topology.

For any $U \in \mathcal{U}$, we denote by $\mathcal{F}_U$ the class of maps $f\colon X \to Y_f$ for which the space Y_f is discrete and the family $\{f^{-1}f(x) \mid x \in X\}$ is U-uniform.

Define the family $\widehat{\mathcal{U}}$ on $F(X)$ as follows. Let $U \in \mathcal{U}$. We write, for $a, b \in F(X)$, $(a, b) \in \widehat{U}$, if there is $f \in \mathcal{F}_U$ such that $F(f)(a) = F(f)(b)$. Let $\widehat{\mathcal{U}} = \{\widehat{U} \mid U \in \mathcal{U}\}$.

Theorem 1. *Let $(X,\mathcal{U})$ be a non-Archimedean uniform space.*

1. *The family $\widehat{\mathcal{U}} = \{\widehat{U} \mid U \in \mathcal{U}\}$ is a non-Archimedean uniform structure on the space $F(X)$.*
2. *If $(X,\mathcal{U}), (Y,\mathcal{V})$ are non-Archimedean uniform spaces and $f\colon X \to Y$ is a uniformly continuous map, then the map $F(f)\colon F(X) \to F(Y)$ is also uniformly continuous.*

Proof. 1) We are going to verify the conditions from the definition of the uniform structure. It is evident that $\Delta_{F(X)} \subset \widehat{U}$, for every $U \in \mathcal{U}$. Then, note that $(\widehat{U})^{-1} = (U^{-1})\widehat{\ }$.

Since the uniformity $\mathcal{U}$ is non-Archimedean, we have $V^2 \subset V$, for every $V \in \mathcal{U}$. We are going to show that then $\widehat{V}^2 \subset \widehat{V}$. Let $(a, b), (b, c) \in \widehat{V}$. Then there exist $f, g \in \mathcal{F}_U$ such that $F(f)(a) = F(f)(b)$ and $F(g)(b) = F(g)(c)$. Consider a couniversal square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y_f \arrow[d, "k"] \\ Y_g \arrow[r, "h"'] & Z \end{tikzcd}$$

and note that $hg = kf \in \mathcal{F}_U$. Indeed, if $hg(x) = hg(y)$, then there exists a finite number of points $x = x_0, x_1, x_2, \dots, x_{2k}, x_{2k+1} = y$ such that

$$f(x_0) = f(x_1),\ g(x_1) = g(x_2),\ f(x_2) = f(x_3), \dots$$

Since $(x_0, x_1), (x_1, x_2), \dots \in V$, we conclude, because of the inclusion $V^2 \subset V$, that $(x, y) \in V$. Then

$$F(kf)(a) = F(kf)(b) = F(hg)(b) = F(hg)(c),$$

whence $(a, c) \in \widehat{V}$.

2) Let $V \in \mathcal{V}$. There exists $U \in \mathcal{U}$ such that $(f \times f)(U) \subset V$. Let $(a, b) \in \widehat{U}$. Then there is $g\colon X \to X_g$, where X_g is a discrete space, such that $F(g)(a) = F(g)(b)$ and the family $\{g^{-1}(y) \mid y \in X_g\}$ is U-uniform. There exists a map $h\colon Y \to Y_h$ such that the family $\{h^{-1}(y) \mid y \in Y_h\}$ is V-uniform and there exists a map $r\colon X_g \to Y_h$ for which the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X_g \arrow[d, "r"] \\ Y \arrow[r, "h"'] & Y_h \end{tikzcd}$$

commutes. Then

$$F(hf)(a) = F(rg)(a) = F(rg)(b) = F(hf)(b),$$

whence $(F(f)(a), F(f)(b)) \in \widehat{V}$. This shows that the map $F(f)\colon F(X) \to F(Y)$ is uniformly continuous. $\square$

We therefore obtain a functor in the category **NA**; we keep the notation F for the obtained functor.

Let $\varphi\colon F \to G$ be a natural transformation of normal functors in the category **COMP**. It is known (see, e.g., [5]) that this natural transformation determines (in a unique way) a natural transformation of the extended functors onto the category **TYCH**. We are going to show that the extended natural transformation generates a natural transformation of the constructed functors in the category **UNIF**.

Proposition 1. *Let $\varphi\colon F \to G$ be a natural transformation of normal functors in the category* **COMP**. *If $(X, \mathcal{U})$ is a non-Archimedean uniform space, then the map $\varphi_X\colon F(X) \to G(X)$ is uniformly continuous (i.e., a morphism in the category* **UNIF***).*

Proof. Since we are dealing with two functors, we have to establish more precise notation. Given $U \in \mathcal{U}$, we let

$$\widehat{U} = \{(a,b) \in F(X) \times F(X) \mid \text{ there exists } f \in \mathcal{F}_U \text{ such that } F(f)(a) = F(f)(b)\},$$
$$\widetilde{U} = \{(a,b) \in G(X) \times G(X) \mid \text{ there exists } f \in \mathcal{F}_U \text{ such that } G(f)(a) = G(f)(b)\}.$$

Let $(a,b) \in \widehat{U}$. Then there exists $f\colon X \to X_f$, $f \in \mathcal{F}_U$, such that $F(f)(a) = F(f)(b)$. Since the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\varphi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(X_f) \arrow[r, "\varphi_{X_f}"'] & G(X_f) \end{tikzcd}$$

is commutative, we conclude that

$$G(f)(\varphi_X(a)) = G(f)(\varphi_X(b))),$$

whence $(\varphi_X(a), \varphi_X(b)) \in \widetilde{U}$. □

All algebraic relations involving natural transformations of normal functors in the category **COMP** are valid also for the corresponding functor in the category **NA**; this allows us to conclude thaty every monad in the category **COMP** generates a monad in the category **NA**.

3.1. Pseudoultrametrics. It is well-known (see, e.g. [10]) that any non-Archimedean uniformity can be generated by a family of pseudo-ultrametrics. In his previous paper [15], the author considered the notion of the normal functor in the category of ultrametric spaces and nonexpanding maps (see, e.g., [7, 8, 21, 1, 19, 2], where different functors are considered in this category). First, we remark that the construction from [15] can be naturally generalized over the case of the category of ultrametric (and, more generally, pseudo-ultrametric) spaces and uniformly continuous maps. Therefore, to every pseudo-ultrametric space (X,d) there corresponds a pseudo-ultrametric space $(F(X), \widehat{d})$. Actually, the two approaches, that of the present paper and the other one based on extensions of pseudo-ultrametrics are equivalent.

3.2. Examples. The first example is that of the power functor $(-)^n$, where n is a natural number. It leads to the standard notion of the product of uniform spaces.

Let $(X,\mathcal{U})$ be a uniform space. On the set $\exp(X)$ of nonempty compact subsets of X the *Hausdorff(-Bourbaki)* uniformity $\mathcal{U}_H$ is generated by the base $\{U_H \mid U \in \mathcal{U}\}$, where

$$U_H = \{(A,B) \mid B \subset U(A) \text{ and } A \subset U(B)\}.$$

Proposition 2. *Let $(X,\mathcal{U})$ be a uniform space. The Hausdorff-Bourbaki uniformity $\mathcal{U}_H$ on $\exp X$ coincides with the uniformity $\widehat{\mathcal{U}}$ which is obtained by the procedure described above (with $F = \exp$).*

Proof. Let $U \in \mathcal{U}$. We may suppose that $U = U^{-1}$. If $(A,B) \in U_H$, then find a finite subset $A_1 \subset A$ such that $\{U(x) \mid x \in A_1\}$ is a disjoint cover of A. Since $B \subset U(A)$, every $U(x)$, $x \in A_1$, meets B. Let $f\colon X \to X$ denote the map defined as follows: $f|(X\setminus U(A_1)) = 1_{X\setminus U(A_1)}$ and $f(x) = y$ if $x \in U(y)$. Then clearly, $f \in \mathcal{F}_U$ and, since $\exp f(A) = f(A) = A_1 = f(B)$, we see that $(A,B) \in \widehat{U}$.

One can similarly prove that $\widehat{U} \subset U_H$, whence the proposition follows. □

4. Remarks and open problems. One can generalize the obtained results in different directions. First, one can consider a wider class of functors than the normal ones, e.g., that of almost normal and weakly normal functors. Recall that a functor F is called *almost normal* (respectively *weakly normal*) if it preserves all the properties from the definition of normal functor but the preimage-preservation (respectively of being epimorphic).

The so called ball structures are introduced by I. Protasov [12] (see also [13]) in order to unify the uniform structures and the coarse structures. To the notion of non-Archimedean uniform structure there corresponds that of cellular ball structure. One can conjecture that there exist counterparts of the normal functors in the category of cellular ball structures.

We leave as an open problem that of extending our considerations onto the case of quasiuniformity. Recall that a quasiuniformity on a set X is a collection of entourages that satisfies the conditions from the definition of uniformity but symmetry. The hyperspace functor in the category of quasiuniform spaces is examined in different publications (see, e.g. [4]).

The questions of completeness of the Hausdorff uniformity on $\exp X$ are investigated in various publications (see, e.g., [11, 9, 3, 20]). We leave as an open problem that of completeness of the uniform spaces $(F(X),\widehat{\mathcal{U}})$, where $(X,\mathcal{U})$ is a complete uniform space.

One of the most intriguing questions is that of completeness of the spaces of probability measures. For general uniform spaces, this question was considered in [6, 14].

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