Topology 48 (2009) 130–136

Contents lists available at ScienceDirect

Topology

journal homepage: www.elsevier.com/locate/top

Fuzzy ultrametrics on the set of probability measures

Aleksandr Savchenko [a,*], Mykhailo Zarichnyi [b]

[a] *Kherson Agrarian University, Kherson, Ukraine*
[b] *Lviv National University, 79000 Lviv, Ukraine*

ARTICLE INFO

MSC:
54E70
54A40
60B05

Keywords:
Fuzzy ultrametric
Fuzzy metric space
Set of probability measures with compact supports

ABSTRACT

We introduce a fuzzy ultrametric on the set of probability measures with compact support defined on a fuzzy metric space. The construction is a counterpart, in the realm of fuzzy ultrametric spaces, of the construction due to Vink and Rutten of an ultrametric on the set of probability measures with compact supports on an ultrametric space.

It is proved that the set of probability measures with finite supports is dense in the natural topology generated by the defined fuzzy ultrametric.

1. Introduction

The notion of fuzzy metric space first appeared in paper [1] and later was modified in [2]. The version from [2], despite being more restrictive, determines the class of spaces that are tightly connected with the class of metrizable topological spaces.

Different notions and results of the theory of metric spaces have their analogues for the fuzzy metric spaces. At the same time, there are phenomena in the realm of fuzzy metric spaces that have no immediate counterparts for metric spaces. The completeness and the existence of non-completable fuzzy metric spaces can serve as an example. This demonstrates that the fuzzy metric seems to be a structure that leads to a theory which seems to be a richer one than that of metric spaces.

The construction of new fuzzy metric spaces out of old ones is of interest in the theory of these spaces. The operation of product of two fuzzy metric spaces is investigated in [3]. In [4], it is shown that there exists a natural fuzzy metric, called the Hausdorff fuzzy metric, on the hyperspace (the set of nonempty compact subsets) of a fuzzy metric space.

There exists an analogue for the notion of ultrametric (non-Archimedean) metric space for fuzzy metric spaces, see [5,6].

The set of probability measures on ultrametric spaces turned out to be an important object of investigations in connection with programming language semantics (see, e.g., [7]). In [8], a natural ultrametric is defined on this set.

It is a natural problem to extend the class of spaces on which the probability measures are compared to that of fuzzy ultrametric spaces. This is done in the present paper.

We consider the notion of fuzzy ultrametric for the case of the t-norm min. Every fuzzy ultrametric is a fuzzy metric and we observe that the Hausdorff fuzzy metric generated by a fuzzy ultrametric is also a fuzzy ultrametric.

The main result is Theorem 4.1, which states that any fuzzy ultrametric on a topological space that induces the topology of this space admits an extension onto the set of probability measures with compact supports in this space.

We also show that the set of measures with finite supports is dense in the obtained space. Finally, we prove that, the map assigning to every probability measure its support, is nonexpanding. Note that this is true for the Vink–Rutten metric but not for the Kantorovich metric on the set of probability measures.

* Corresponding author.
E-mail addresses: savchenko1960@rambler.ru (A. Savchenko), mzar@litech.lviv.ua (M. Zarichnyi).

0040-9383/$ – see front matter
doi:10.1016/j.top.2009.11.011

2. Preliminaries

2.1. Fuzzy metric spaces

The notion of fuzzy metric space, in one of its forms, is introduced by Kramosil and Michalek [1]. In the present paper we use the version of this concept given in the paper [2] by George and Veeramani.

Definition 2.1. A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0, 1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0, 1]$.

The following are examples of t-norms: $a * b = ab$; $a * b = \min\{a, b\}$.

Definition 2.2. A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(v) the function $M(x, y, -): (0, \infty) \to [0, 1]$ is continuous.

It is proved in [2] that in a fuzzy metric space X, the function $M(x, y, -)$ is nondecreasing for all $x, y \in X$.
The following notion is introduced in [2] (see Definition 2.6 therein).

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set

$$B(x, r, t) = \{y \in X \mid M(x, y, t) > 1 - r\}$$

is called the *open ball* with center x and radius r with respect to t.

The family of all open balls in a fuzzy metric space $(X, M, *)$ forms a base of a topology in X; this topology is denoted by τ_M and is known to be metrizable (see [2]).
If $(X, M, *)$ is a fuzzy metric space and $Y \subset X$, then, clearly,

$$M_Y = M \mid (Y \times Y \times (0, \infty)): Y \times Y \times (0, \infty) \to [0, 1]$$

is a fuzzy metric on the set Y. We say that the fuzzy metric M_Y is *induced* on Y by M.
Let $(X, M, *)$ and $(X', M', *)$ be fuzzy metric spaces. A map $f: X \to X'$ is called *nonexpanding* if $M'(f(x), f(y), t) \ge M(x, y, t)$, for all $x, y \in X$ and $t > 0$. For our purposes, it is sufficient to consider the class of fuzzy metric spaces with the same fixed norm (e.g., $*$). The fuzzy metric spaces (with the norm $*$) and nonexpanding maps form a category, which we denote by $\mathcal{FMS}(*)$.

2.2. Spaces of probability measures

Let X be a metrizable space. By $P(X)$ we denote the space of probability measures with compact support in X (see, e.g., [9] for the necessary definitions that concern probability measures). Recall that the *support* of a probability measure $\mu \in P(X)$ is the minimal (with respect to the inclusion) closed set $\mathrm{supp}(\mu)$ such that $\mu(X \setminus \mathrm{supp}(\mu)) = 0$. For any $x \in X$, by δ_x we denote the Dirac measure concentrated at x.
Any probability measure μ of finite support can be represented as follows: $\mu = \Sigma_{i=1}^n \alpha_i \delta_{x_i}$, where $\alpha_1, \ldots, \alpha_n \ge 0$ and $\Sigma_{i=1}^n \alpha_i = 1$. By $P_\omega(X)$ we denote the set of all probability measures with finite supports in X.
Every continuous map $f: X \to Y$ of metrizable spaces generates a map $P(f): P(X) \to P(Y)$ defined by the condition: $P(f)(\mu)(A) = \mu(f^{-1}(A))$, for every Borel subset of Y.

3. Fuzzy ultrametric spaces

One can define a counterpart of the notion of ultrametric in the realm of fuzzy metric spaces (see, e.g., [6]).

Definition 3.1. A 3-tuple $(X, M, *)$ is said to be a fuzzy ultrametric space if X is an arbitrary set, $* = \min$ and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying conditions (i), (ii), (iii), (v) from Definition 2.2 and the following condition:

(iv′) $M(x, y, t) * M(y, z, s) \le M(x, z, \max\{t, s\})$, for all $x, y, z \in X$ and $t, s \in (0, \infty)$.

In [6], it is remarked that condition (iv′) is equivalent to the following:

(iv″) $M(x, y, t) * M(y, z, t) \le M(x, z, t)$, for all $x, y, z \in X$ and $t \in (0, \infty)$

(see [5, Definition 5]).

Lemma 3.2. *Let M be a fuzzy ultrametric on a set X. Then, for every $x, y \in X$, the function $M(x, y, -): (0, \infty) \to [0, 1]$ is nondecreasing.*

Proof. Let $s \le t$. Then

$$M(x, y, s) = \min\{M(x, y, s), 1\} \le \min\{M(x, y, s), M(y, y, t)\} \le M(x, y, t). \quad \square$$

Remark 3.3. In the proof of Lemma 3.2, we did not use property (v) from the definition of the fuzzy ultrametric.

Proposition 3.4. *Every fuzzy ultrametric is a fuzzy metric.*

Proof. Given $t, s \in (0, 1)$ and $x, y \in X$, we obtain, by Lemma 3.2,

$$\min\{M(x, y, t), M(y, z, s)\} \le M(x, z, \max\{t, s\}) \le M(x, z, t + s). \quad \square$$

Let $\varphi: (0, \infty) \to (0, 1)$ be any continuous increasing surjection. Suppose that d is an ultrametric on a set X such that $d(x, y) < 1$ for all $x, y \in X$. Define $M_d: X^2 \times (0, \infty) \to [0, 1]$ as follows:

$$M_d(x, y, t) = 1 - d(x, y) + d(x, y)\varphi(t).$$

Proposition 3.5. *The function M_d is a fuzzy ultrametric on X.*

Proof. Conditions (i)–(iii) and (v) from the definition of fuzzy (ultra)metric are obviously satisfied.

We have to verify condition (iv′). Let $x, y, z \in X$, $t, s \in (0, 1)$. Without loss of generality, we may assume that $t \ge s$. The proof splits in two cases.

(1) $d(x, y) = d(x, z)$. Then

$$\begin{aligned}\min\{M_d(x, y, t), M_d(y, z, s)\} &= \min\{1 - d(x, y) + d(x, y)\varphi(t), 1 - d(y, z) + d(y, z)\varphi(s)\}\\ &\le 1 - d(x, z) + d(x, z)\varphi(t) = M_d(x, z, \max\{t, s\}).\end{aligned}$$

(2) $d(y, z) = d(x, z)$. Then

$$\begin{aligned}\min\{M_d(x, y, t), M_d(y, z, s)\} &= \min\{1 - d(x, y) + d(x, y)\varphi(t), 1 - d(y, z) + d(y, z)\varphi(s)\}\\ &\le 1 - d(y, z) + d(y, z)\varphi(s) \le 1 - d(y, z) + d(y, z)\varphi(t) = M_d(x, z, \max\{t, s\}). \quad \square\end{aligned}$$

For any Hausdorff topological space X, we denote by $\exp X$ the set of all nonempty compact subsets of X.

Proposition 3.6. *Let $(X, M, *)$ be a fuzzy ultrametric space. Then the Hausdorff fuzzy metric space $(\exp X, M_H, *)$ is a fuzzy ultrametric space.*

Proof. We use the following definition of the Hausdorff metric M_H on the set $\exp X$, which is equivalent to the initial one. If $C, D \in \exp X$ and $t \in (0, \infty)$, we let

$$M_H(C, D, t) = 1 - \inf\{r \mid C \subset B(D, r, t),\ D \subset B(C, r, t)\}.$$

Let $A, C, D \in \exp X$ and $s, t \in (0, \infty)$. We have to show that

$$M_H(A, C, s) * M_H(C, D, t) \le M_H(A, D, \max\{t, s\}).$$

Without loss of generality, we assume that $s \le t$.

Suppose that $M_H(A, C, s) > 1 - r$, $M_H(C, D, t) > 1 - r$, for some r. Then also $M_H(A, C, t) > 1 - r$. Given $a \in A$, we see that there exists $c \in C$ such that $M(a, c, t) > 1 - r$. Similarly, there is $d \in D$ such that $M(c, d, t) > 1 - r$. Then

$$M(a, d, t) \ge \min\{M(a, c, t), M(c, d, t)\} > 1 - r,$$

and therefore $A \subset B(D, r, t)$. One can similarly show that $D \subset B(A, r, t)$, whence $M_H(A, D, t) > 1 - r$ and we are done. $\square$

Let $(X, M, *)$ be a fuzzy metric space. Given $c \in (0, 1]$, define

$$c \odot M(x, y, t) = 1 - c + cM(x, y, t), \quad (x, y, t) \in X \times X \times (0, \infty).$$

Lemma 3.7. *For any fuzzy (ultra)metric M on X, the function $c \odot M$ is a fuzzy (ultra)metric on X. The topologies generated by M and $c \odot M$ coincide.*

Proof. The verification that $c \odot M$ is a fuzzy (ultra)metric, for any fuzzy (ultra)metric M, is easy and is left to the reader.

Let us use subscript c to denote the balls with respect to the fuzzy (ultra)metric $c \odot M$. Then we have

$$\begin{aligned} B_c(x, r, t) &= y \in X \mid c \odot M(x, y, t) > 1 - r \\ &= y \in X \mid 1 - c + cM(x, y, t) > 1 - r = y \in X \mid M(x, y, t) > 1 - (r/c) \\ &= B(x, r/c, t), \end{aligned}$$

if $r \le c$, and $B_c(x, r, t) = X$ otherwise, whence the coincidence of the topologies follows. □

Let $(X, M, *)$ be a fuzzy metric space. Let $\mathbb{N}$ denote, as usual, the set of positive integers. By $X^{\mathbb{N}}$ we denote the countable power of X. Define $\bar{M}: X^{\mathbb{N}} \times X^{\mathbb{N}} \times (0, \infty) \to \mathbb{R}$ as follows:

$$\bar{M}((x_i), (y_i), t) = \inf\{(1/i) \odot M(x_i, y_i, t) \mid i \in \mathbb{N}\}.$$

Lemma 3.8. *For any fuzzy (ultra)metric M on X, the function $\bar{M}$ is a fuzzy (ultra)metric on $X^{\mathbb{N}}$. The topology generated by $\bar{M}$ is the product topology on $X^{\mathbb{N}}$ (the topology on each factor is generated by M).*

Proof. Let us verify property (v) from the definition of fuzzy (ultra)metric. Let $t_0 \in (0, \infty)$. Let $(x_i), (y_i) \in X^{\mathbb{N}}$. There is $i_0 \in \mathbb{N}$ and a neighborhood U of t_0 in $(0, \infty)$ such that $M(x_1, y_1, t) > 1 - (1/i_0)$, for all $t \in U$. Then, for any $t \in U$,

$$\bar{M}((x_i), (y_i), t) = \min\{(1/i) \odot M(x_i, y_i, t) \mid i < i_0\},$$

therefore $\bar{M}$ is continuous on U.

The verification of the remaining properties from the definition of fuzzy (ultra)metric is straightforward.

Since

$$\bar{B}((x_i), r, t) = \prod_{i \in \mathbb{N}} B_{\frac{1}{i}}(x_i, r, t)$$

(here, $\bar{B}$ stands for the ball with respect to the fuzzy metric $\bar{M}$) and

$$B_{\frac{1}{i}}(x_i, r, t) = X$$

whenever $\frac{1}{i} < r$, we conclude that the topology on $X^{\mathbb{N}}$ generated by $\bar{M}$ coincides with the product topology. □

Lemma 3.9. *Let $x, y \in X$, $r \in (0, 1)$, and $t > 0$. If $B(x, r, t) \cap B(y, r, t) \ne \emptyset$, then $B(x, r, t) = B(y, r, t)$.*

Proof. Let $z \in B(x, r, t) \cap B(y, r, t)$. Then

$$1 - r < \min\{M(x, z, t), M(y, z, t)\} \le M(x, y, t),$$

whence $y \in B(x, r, t)$.

Now if $z \in B(y, r, t)$, then similarly

$$1 - r < \min\{M(x, y, t), M(y, z, t)\} \le M(x, z, t),$$

whence $z \in B(x, r, t)$. We conclude that $B(y, r, t) \subset B(x, r, t)$. Analogously, $B(y, r, t) \supset B(x, r, t)$. □

Corollary 3.10. *Every ball $B(x, r, t)$ is an open and closed set (i.e., a clopen set).*

Recall that a topological space is called *zero dimensional* if there is a base for the topology of this space consisting of clopen sets.

The following statement is a counterpart for fuzzy ultrametric spaces of that of the zero dimensionality of ultrametric spaces.

Proposition 3.11. *Every fuzzy ultrametric space is zero dimensional.*

Proposition 3.12. *Suppose that $t \ge s$. Then every open r-ball for t is a union of open r-balls for s.*

Proof. Let $y \in B(x, r, t)$ and $z \in B(y, r, s)$. Then $M(x, y, t) > 1 - r$ and $M(y, z, s) > 1 - r$ and we obtain

$$1 - r < \min\{M(x, y, t), M(y, z, s)\} \le M(x, z, \max\{t, s\}) = M(x, z, t),$$

whence $z \in B(x, r, t)$. We have proven that $B(y, r, s) \subset B(x, r, t)$, whence the result follows. □

In [10], the following uniform structure is defined for a fuzzy metric space $(X, M, *)$:

$$\mathcal{U} = \{\{(x, y) \mid x \in B(y, 1/n, 1/n)\} \mid n \in \mathbb{N}\}.$$

Recall that a uniform space is of *uniform dimension* $\le n$ (see, e.g., [11]), if each of its uniform covers admits a refinement of dimension $\le n$.

Proposition 3.13. *Any fuzzy ultrametric space is of uniform dimension zero.*

Lemma 3.14. *Let A be a compact subset of X, $t_0 \in (0, \infty)$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $x, y \in A$ and $|t - t_0| < \delta$, then $|M(x, y, t) - M(x, y, t_0)| < \varepsilon$.*

Proof. It is known (see [4, Proposition 1]) that the map $M: X \times X \times (0, \infty) \to [0, 1]$ is continuous. Therefore, the restriction $M|(A \times A \times J)$, where J is a segment in $(0, \infty)$ containing t_0 is uniformly continuous, whence the assertion follows. □

3.1. Extension of fuzzy ultrametrics

Theorem 3.15. *Let X be a fuzzy separable zero-dimensional metrizable space, A a closed subset in X with $|A| \geq 2$. Let M be a fuzzy ultrametric on A compatible with its topology. Then there exists a fuzzy ultrametric M' on X that extends M and generates the topology on X.*

Proof. Let $\bar{M}$ be a fuzzy ultrametric on $A^{\mathbb{N}}$ defined by the formula:

$$\bar{M}((x_i), (y_i), t) = \inf\{(1/i) \odot M(x_i, y_i, t) \mid i \in \mathbb{N}\}.$$

There exists an embedding $j: X \to A^{\mathbb{N}}$ such that $j(a) = (a, a, a, \ldots)$, for every $a \in A$.

Define $M': X \times X \times (0, \infty) \to \mathbb{R}$ by the formula:

$$M'(x, y, t) = \bar{M}(j(x), j(y), t).$$

Clearly, M' is a fuzzy ultrametric on X generating its topology. Since, for every $x, y \in A$, we have

$$\begin{aligned} M'(x, y, t) &= \bar{M}((x, x, x, \ldots), (y, y, y, \ldots), t) \\ &= \inf\{(1/i) \odot M(x, y, t) \mid i \in \mathbb{N}\} = M(x, y, t), \end{aligned}$$

and therefore M' is an extension of M.

Since j is an embedding, M' generates the topology of X. □

4. Main result

Given a fuzzy ultrametric space $(X, M, \min)$, define a function $\hat{M}: P(X) \times P(X) \times (0, \infty) \to [0, 1]$ by the formula:

$$\hat{M}(\mu, \nu, t) = 1 - \inf\{r \in (0, 1) \mid \mu(B(x, r, t)) = \nu(B(x, r, t)), \text{ for every } x \in X\}.$$

Note that, one can only require that $x \in \operatorname{supp}(\mu) \cup \operatorname{supp}(\nu)$ in this formula.

Theorem 4.1. *The function $\hat{M}$ is a fuzzy ultrametric on the set $P(X)$ (with respect to the t-norm min).*

Proof. Conditions (i) and (iii) from Definition 2.2 are obviously satisfied.

Let us verify condition (ii). Clearly, $\hat{M}(\mu, \mu, t) = 1$, for every $\mu \in P(X)$ and $t > 0$. Conversely, if $\hat{M}(\mu, \nu, t) = 1$, then $\mu(B(x, r, t)) = \nu(B(x, r, t))$, for every $x \in X$, $r \in (0, 1)$ and $t > 0$. Because of σ-additivity, we conclude that the values of μ and ν coincide on all Borel subsets of X. In turn, this implies that $\mu = \nu$.

Let us verify Condition (iv′) from Definition 3.1. Let $\mu, \nu, \tau \in P(X)$, $t, s \in (0, \infty)$. Suppose, without loss of generality, that $\max\{t, s\} = t$.

If $\hat{M}(\mu, \nu, t) > 1 - r$ and $\hat{M}(\nu, \tau, s) > 1 - r$, then $\mu(B(x, r, t)) = \nu(B(x, r, t))$, for all $x \in X$. Also, $\nu(B(x, r, s)) = \tau(B(x, r, s))$, for all $x \in X$, and, since every ball of radius r for t is the union of disjoint family of balls of radius r for s (see Proposition 3.12), we conclude, because of the additivity of the measures, that $\nu(B(x, r, t)) = \tau(B(x, r, t))$, for all $x \in X$.

Therefore,

$$\mu(B(x, r, t)) = \nu(B(x, r, t)) = \tau(B(x, r, t)),$$

for all $x \in X$, whence $\hat{M}(\mu, \tau, t) > 1 - r$ and the result follows.

We are now going to verify condition (v) from Definition 2.2. Let $\mu, \nu \in P(X)$, $t_0 \in (0, \infty)$ and $\hat{M}(\mu, \nu, t_0) = 1 - r_0$.

Let $(t_i)_{i=1}^{\infty}$ be a nondecreasing sequence in $(0, \infty)$ with $\lim_{i\to\infty} t_i = t_0$. Let $\hat{M}(\mu, \nu, t_i) = 1 - r_i$, $i = 0, 1, 2, \ldots$. Then $(r_i)_{i=1}^{\infty}$ is a nonincreasing sequence in $(0, 1]$ (see Remark 3.3). Suppose that $r_0' = \lim_{i\to\infty} r_i > r_0 + 2c$, for some $c > 0$.

Then $\mu(B(x, r_0 + c, t_0)) = \nu(B(x, r_0 + c, t_0))$, for all $x \in X$. By Lemma 3.14, there exists $\eta > 0$ such that, for every $x, y \in \operatorname{supp}(\mu) \cup \operatorname{supp}(\nu)$, we have

$$|M(x, y, t_0 - \eta) - M(x, y, t_0)| < c.$$

There exists $i \in \mathbb{N}$ such that $t_i > t_0 - \eta$. We are going to show that

$$B(x, r_0, t_0) \cap (\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu)) \subset B(x, r_0 + c, t_i).$$

Indeed, if $y \in B(x, r_0, t_0) \cap (\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu))$, then $M(x, y, t_0) > 1 - r_0$, whence $M(x, y, t_i) > 1 - r_0 - c$ and therefore $y \in B(x, r_0 + c, t_i)$. Now, if $y \in B(x, r_0, t_0) \cap (\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu))$, then $y \in B(y, r_0, t_0) = B(x, r_0, t_0) \subset B(x, r_0 + c, t_i)$.

We therefore conclude that every set $B(x, r_0 + c, t_i) \cap (\text{supp}(\mu) \cup \text{supp}(\nu))$, where $x \in \text{supp}(\mu) \cup \text{supp}(\nu)$ is a disjoint union of the sets of the form $B(y, r_0, t_0) \cap (\text{supp}(\mu) \cup \text{supp}(\nu))$, where $y \in \text{supp}(\mu) \cup \text{supp}(\nu)$. Since μ and ν are additive set functions, we have

$$\begin{aligned}\mu(B(x, r_0 + c, t_i)) &= \mu(B(x, r_0 + c, t_i) \cap (\text{supp}(\mu) \cup \text{supp}(\nu)))\\ &= \nu(B(x, r_0 + c, t_i) \cap (\text{supp}(\mu) \cup \text{supp}(\nu))) = \nu(B(x, r_0 + c, t_i)),\end{aligned}$$

for all $x \in \text{supp}(\mu) \cup \text{supp}(\nu)$. Therefore, $\hat{M}(\mu, \nu, t_i) \leq 1 - (r_0 + c) < 1 - r'_0$, and we obtain a contradiction with the assumption $r_0 < r'_0$.

Next, we consider the case of a nonincreasing sequence $(t_i)_{i=1}^{\infty}$ in $(0, \infty)$ with $\lim_{i\to\infty} t_i = t_0$. Then $(r_i = 1 - \hat{M}(\mu, \nu, t_i))_{i=1}^{\infty}$ is a nondecreasing sequence in $(0, 1]$ (see Remark 3.3). Suppose that $r'_0 = \lim_{i\to\infty} r_i < r_0$. There exists $c > 0$ such that $r'_0 + 2c < r_0$.

By Lemma 3.14, there exists $\eta > 0$ such that, for every $x, y \in \text{supp}(\mu) \cup \text{supp}(\nu)$, we have

$$|M(x, y, t_0 + \eta) - M(x, y, t_0)| < c.$$

There exists $i \in \mathbb{N}$ such that $t_i < t_0 + \eta$. Arguing as above, we conclude that

$$B(x, r'_0, t_i) \cap (\text{supp}(\mu) \cup \text{supp}(\nu)) \subset B(x, r'_0 + c, t_0).$$

This, in turn, implies that $\hat{M}(\mu, \nu, t_0) \geq 1 - (r'_0 + c) > 1 - r_0$ and we obtain a contradiction. □

Identifying every $x \in X$ with the Dirac measure δ_x, one may regard X as a subset of $P(X)$.

Proposition 4.2. *Let $(X, M, *)$ be a fuzzy ultrametric space. Then the fuzzy ultrametric $\hat{M}$ induces the fuzzy ultrametric M on $X \subset P(X)$.*

Proof. Let $x, y \in X$. If $\hat{M}(\delta_x, \delta_y, t) = 1 - r$, then, for any $r' > r$, we have $1 = \delta_x(B(x, r', t)) = \delta_y(B(x, r', t))$, whence $y \in B(x, r', t)$ and therefore $M(x, y, t) > 1 - r'$. Passing to the limit as $r' \to r$, we see that $M(x, y, t) \geq \hat{M}(\delta_x, \delta_y, t)$.

On the other hand, if $r' < r$, then there is $z \in X$ such that $\delta_x(B(z, r', t)) \neq \delta_y(B(z, r', t))$. Without loss of generality, one may assume that

$$1 = \delta_x(B(z, r', t)).$$

Then $y \notin B(z, r', t) = B(x, r', t)$, whence $M(x, y, t) \leq 1 - r'$. Passing to the limit as $r' \to r$, we see that $M(x, y, t) \leq \hat{M}(\delta_x, \delta_y, t)$. □

Proposition 4.3. *The set $P_\omega(X)$ is dense in $P(X)$ in the topology induced by the fuzzy ultrametric $\hat{M}$.*

Proof. Let $\mu \in P(X)$, $r \in (0, 1)$, $t > 0$. Consider an open cover $\mathcal{U} = \{B(x, r, t) \mid x \in \text{supp}(\mu)\}$ of the set $\text{supp}(\mu)$. Since the set $\text{supp}(\mu)$ is compact, there exists a finite subcover $\{B(x_i, r, t) \mid i = 1, \ldots, k\}$ of $\mathcal{U}$. Let $\nu = \sum_{i=1}^{k} \mu(B(x_i, r, t))\delta_{x_i}$. Then $\nu \in P_\omega(X)$.

We are going to show that $\hat{M}(\mu, \nu, t) > 1-r$. To this end, consider $B(z, r, t)$, for $z \in X$. If $z \in \cup\{B(x_i, r, t) \mid i = 1, \ldots, k\}$, then, using Lemma 3.9 one can suppose that $z = x_i$, for some $i = 1, \ldots, k$. Then

$$\nu(B(z, r, t)) = \mu(B(z, r, t)) = \mu(B(x_i, r, t)).$$

If $z \notin \cup\{B(x_i, r, t) \mid i = 1, \ldots, k\}$, then, using Lemma 3.9 again, one can show that also

$$B(z, r, t) \cap \cup\{B(x_i, r, t) \mid i = 1, \ldots, k\} = \emptyset,$$

whence $\nu(B(z, r, t)) = \mu(B(z, r, t)) = 0$.

We conclude that $\hat{M}(\mu, \nu, t) > 1 - r$ and therefore $\nu \in B(\mu, r, t)$. □

Proposition 4.4. *The map* supp: $P(X) \to \exp X$ *is nonexpanding.*

Proof. Let $\mu, \nu \in P(X)$ and $\hat{M}(\mu, \nu, t) > r_0$, where $r_0 \in (0, 1)$. Then from the definition of $\hat{M}$ it follows that there exists $r < 1 - r_0$ such that $\nu(B(x, r, t)) = \mu(B(x, r, t))$, for all $x \in X$.

Suppose that $z \in \text{supp}(\mu)$, then from the definition of support it follows that $\nu(B(z, r, t)) = \mu(B(z, r, t)) > 0$ and therefore there exists $z' \in \text{supp}(\nu)$ such that $z' \in B(z, r, t)$.

Therefore, $\text{supp}(\mu) \subset B(\text{supp}(\nu), r, t)$. One can similarly show that $\text{supp}(\nu) \subset B(\text{supp}(\mu), r, t)$.

This implies that

$$M_H(\text{supp}(\mu), \text{supp}(\nu), t) > 1 - r > 1 - (1 - r_0) = r_0$$

and we conclude that the map supp is nonexpanding. □

Proposition 4.5. *Let $(X, M, *)$, $(X', M', *)$ be fuzzy ultrametric spaces and let $f: X \to X'$ be a nonexpanding map. Then the map $P(f): P(X) \to P(X')$ is also nonexpanding.*

Proof. We are going to show that, for every $\mu, \nu \in P(X)$ and $t > 0$, if $\hat{M}(\mu, \nu, t) > \varrho$ then $\hat{M}'(P(f)(\mu), P(f)(\nu), t) > \varrho$.

Given $\hat{M}(\mu, \nu, t) > \varrho$, one can find $r \in (0, 1)$ such that $1 - r > \varrho$ and $\mu(B(x, r, t)) = \nu(B(x, r, t))$, for all $x \in X$. Since the map f is nonexpanding, we see that $f(B(x, r, t)) \subset B'(f(x), r, t)$ (by B' we denote the balls in X'), whence, by Lemma 3.9, for every $y \in X'$, the set $f^{-1}(B'(y, r, t))$ is a union of disjoint balls of the form $B(z, r, t)$ in X. Therefore, from the σ-additivity of μ and ν, it follows that

$$\begin{aligned} P(f)(\mu)(B'(y, r, t)) &= \mu(f^{-1}(B'(y, r, t))) \\ &= \nu(f^{-1}(B'(y, r, t))) = P(f)(\nu)(B'(y, r, t)), \end{aligned}$$

for all $y \in X'$, whence $\hat{M}'(P(f)(\mu), P(f)(\nu), t) > \varrho$. □

It is easy to see that from Proposition 4.5 it follows that P is a functor from the category $\mathcal{FMS}(*)$ to itself. One can also prove the same for the hyperspace construction exp. Then, from Proposition 4.4 one can deduce that supp is a natural transformation of the functor P into the functor exp.

5. Remarks and open questions

In [12] the functor of idempotent measures in the category of ultrametric spaces is defined. We conjecture that a counterpart of this functor can be defined also for fuzzy ultrametric spaces.

In [4] it is proved that the hyperspace of a fuzzy metric space is complete if and only if the space itself is complete. Recall (see [2]) that a fuzzy metric space $(X, M, *)$ is *complete* provided that every Cauchy sequence in X is convergent with respect to the topology τ_M, where a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be *Cauchy* if for each $r \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ for all $n, m > n_0$. We leave as an open problem that of completeness (as well as completability) of the fuzzy ultrametric spaces of probability measures considered in this paper.

In [13], a notion of intuitionistic fuzzy metric space is introduced and studied. See [14] for connections between fuzzy metric spaces and intuitionistic fuzzy metric spaces. In particular, in [14], an intuitionistic fuzzy metric is considered on the hyperspace of an intuitionistic metric space. We formulate the problem of existence of an intuitionistic fuzzy metric on the set of probability measures on intuitionistic fuzzy metric spaces.

It is interesting to compare the weak* topology on the set $P(X)$ and the topology generated by the fuzzy ultrametric.

In connection with Theorem 3.15, the following two questions arise:

Question 5.1. Can one drop the condition of separability of the space X?

Question 5.2. Is there a counterpart of this theorem for the uniform spaces?

We finish with the problem of proving similar results for the idempotent measures (see [12]).

Acknowledgement

The authors are sincerely indebted to the referee for the careful reading of the manuscript and numerous remarks and corrections.

References

[1] O. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 326–334.
[2] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (1997) 365–368.
[3] Mohd. Rafi Segi Rahmat, Mohd. Salmi Md. Noorani, Product of fuzzy metric spaces and fixed point theorems, International Journal of Contemporary Mathematical Sciences 3 (15) (2008) 703–712.
[4] J. Rodríguez-López, S. Romaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy Sets and Systems 147 (2) (2004) 273–283.
[5] S. Romaguera, A. Sapena, P. Tiradoi, The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words, Topology and its Applications 154 (2007) 2196–2203.
[6] D. Miheţ, Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems Archive 159 (6) (2008) 739–744.
[7] J.W. de Bakker, J.I. Zucker, Processes and denotational semantics of concurrency, Information and Control 54 (1982) 70–120.
[8] E.P. de Vink, J.J.M.M. Rutten, Bisimulation for probabilistic transition systems: A coalgebraic approach, Theoretical Computer Science 221 (1–2) (1999) 271–293.
[9] K.R. Parthasarathy, Probability Measures on Metric Spaces, Academic Press, 1967.
[10] V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets and Systems 115 (2000) 485–489.
[11] J.R. Isbell, Uniform Spaces, in: Mathematical Surveys, nr. 12, AMS, Providence, Rhode Island, 1964.
[12] O. Hubal, M. Zarichnyi, Idempotent probability measures on ultrametric spaces, Journal of Mathematical Analysis and Applications 343 (2008) 1052–1060.
[13] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 22 (2004) 1039–1046.
[14] V. Gregori, S. Romaguera, P. Veeramani, A note on intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 28 (4) (2006) 902–905.