Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
17th Conference on Applied Mathematics **APLIMAT 2018**

Proceedings

MAPPINGS OF SPACES WITH AFFINE CONNECTION

KIOSAK Volodymyr (UA), LESECHKO Olexandr (UA), SAVCHENKO Olexandr (UA)

Abstract. By modification of A. Norden methodic, we have found the formulae connecting main tensors of spaces with affine connection A_n and $\bar{A}_n$, admitting mappings onto each other. We introduce the notion of shortened mapping and its particular case a half-mapping. When we turn our attention to covariant derivatives under medium connection, the main equations are simplified to a notable degree.

Keywords: mappings, spaces with affine connection, deformation tensor, Riemannian tensor

Mathematics subject classification: Primary 53B30, 53B35

1 Introduction

The notion of a connection was introduced to differential geometry in works of Tullio Levi-Civita (1917). Later H. Weyl (1920) invoked the spaces with affine connection while studying translation of vectors.

This paper treats the mappings of spaces with affine connection. Morphisms (mappings) of generalized geometric spaces are a subject of an up-to-date research field in modern differential geometry. There are three main directions of inquiry [1, 2, 3, 4]:

1. A study of general patterns of mappings.
2. Research on the problem whether a given generalized space admits a special mapping or not.
3. Search for a connecting mapping when a pair of spaces is given.

The scheme of investigation is as follows: study of mappings is reduced to a system of differential equations; the system of differential equations is reduced to an algebraic system, which represents conditions of integrability; these systems redefined by introduction of additional limitations, they are simplified or integrated [5].

A local solution of this problem is possible in general, meanwhile in practice it meets serious technical difficulties.

The paper suggests the method for construction of a simplified system of differential equations in covariant derivatives for research on mappings of spaces with affine connection.

2 Main results

A space with affine connection A_n of a given dimension n is a differential manifold with every curve having an affine connection defined. Or in other words: for any point M and for any vector field in the vicinity of this point, absolute differential of a vector belonging to this field, if calculated in the point M for any curve passing through it, is a linear function of a vector of elementary translation along the curve.

Authors treat spaces with affine connections A_n without torsion, as follows

$$\Gamma_{ij}^h(x) = \Gamma_{ji}^h(x). \tag{1}$$

The space A_n belongs to class C^r $(A_n \in C^r)$, if $\Gamma_{ij}^h(x) \in C^r$.

Here we treat two spaces with affine connection.

A one-to-one correspondence of points of spaces with affine connection A_n and $\bar{A}_n$ is called a mapping. Then in a system of coordinates common in respect to the mapping the following conditions exist:

$$\bar{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x) = P_{ij}^h(x). \tag{2}$$

here $\Gamma_{ij}^h, \bar{\Gamma}_{ij}^h$ — objects with affine connection of spaces A_n and $\bar{A}_n$ respectively. In the following discussion the objects $\bar{A}_n$ will be denoted by a bar.

A system of curvilinear coordinates is called a system of coordinates common in respect to a mapping if the coordinates of respective points coincide.

The tensor $P_{ij}^h(x)$ is called a tensor of deformation of connection at a given mapping.

If

$$P_{ij}^h(x) \not\equiv 0, \tag{3}$$

then the mapping is called nontrivial.

Let us mention that the deformation tensor is symmetrical in sense of covariant indexes. Or $P_{ij}^h = P_{ji}^h$ for torsion-free spaces with affine connection.

Theorem 1. *When the space with affine connection A_n is mapped onto the space with affine connection $\bar{A}_n$, Riemannian tensors of spaces A_n and $\bar{A}_n$ are connected by an equation in a single coordinate system*

$$\bar{R}^h_{\cdot\, ijk} = R^h_{\cdot\, ijk} + \frac{1}{2}(\nabla_k P_{ji}^h - \nabla_j P_{ki}^h + \bar{\nabla}_k P_{ji}^h - \bar{\nabla}_j P_{ki}^h). \tag{4}$$

HereR^h_{ijk} — Riemannian tensor and ∇ — a symbol of covariant derivative.

While for purposes of definition of mapping, we introduce a one-to-one correspondence, in fact we order the given pair of spaces with affine connection A_n and $\bar{A}_n$ by ascribing a sign for a deformation tensor. Any pair of spaces A_n and $\bar{A}_n$ has a correspondence defined by objects of connections of these spaces. On the other hand, object of connection A_n and deformation tensor characterize connection of space $\bar{A}_n$. It permits to introduce a mapping that we propose to name shortened in relation to a given mapping. Further we will call it a shortened mapping.

Object $\overset{\lambda}{\Gamma}{}^h_{ij}$ is constructed following a rule

$$\overset{\lambda}{\Gamma}{}^h_{ij} = \Gamma^h_{ij}(x) + \frac{\lambda}{1+\lambda} P^h_{ij}(x), \qquad (5)$$
$$\lambda = const > 0.$$

It characterizes the connection of a given space with affine connection $\overset{\lambda}{A}_n$ [6].

A mapping of space with affine connection A_n onto a space with affine connection $\overset{\lambda}{A}_n$ is called a shortened mapping, when such an equation is true in a common system of coordinates (5).

Theorem 2. *When spaces A_n and $\bar{A}_n$ admit a mapping that corresponds to deformation tensor P^h_{ij}, then there exists a shortened mapping, in respect to which Riemannian tensor corresponds to limitations:*

$$\overset{\lambda}{R}{}^h_{.\,ijk} = R^h_{.\,ijk} + \frac{\lambda}{1+\lambda}(\nabla_k P^h_{ji} - \nabla_j P^h_{ki} + \overset{\lambda}{\nabla}_k P^h_{ji} - \overset{\lambda}{\nabla}_j P^h_{ki}). \qquad (6)$$

When $\lambda = 1$ then such a mapping is called shortened in half or a half-mapping and the connection is called medium.

Theorem 3. *When spaces A_n and $\bar{A}_n$ admit the mapping that corresponds to the deformation tensor P^h_{ij}, then there exists the half-mapping with Riemannian tensor that satisfies conditions:*

$$\bar{R}^h_{.\,ijk} = R^h_{.\,ijk} + \overset{c}{\nabla}_k P^h_{ji} - \overset{c}{\nabla}_j P^h_{ki}. \qquad (7)$$

Contracting the latter with respect to indices h and i, we get

$$\bar{R}^\alpha_{\alpha jk} = R^\alpha_{\alpha jk} + \overset{c}{\nabla}_k \tau_j - \overset{c}{\nabla}_j \tau_k, \qquad (8)$$

where $\tau_i \overset{def}{=} P^\alpha_{i\alpha} = \bar{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha}$.

A manifold A_n with a symmetric affine connection is called equiaffine manifold if the Ricci tensor is symmetric.

Since in the case of symmetric connection

$$R_{ij} - R_{ji} = R^\alpha_{\alpha ji}. \qquad (9)$$

A manifold A_n with a symmetric affine connection is an equiaffine manifold if and only if in any coordinate system (x^i) there exists a function $f(x)$ satisfying

$$\Gamma^\alpha_{i\alpha} = \partial_i f(x). \qquad (10)$$

When the spaces are equiaffine and the Ricci tensors are symmetrical then we get the equation:

$$\overset{c}{\nabla}_k \tau_j - \overset{c}{\nabla}_j \tau_k = 0. \tag{11}$$

Let us formulate the theorem:

Theorem 4.

If the Weyl tensor is preserved in the course of the mapping of spaces with affine connection, the deformation tensor holds the conditions:

$$\begin{gathered}\overset{c}{\nabla}_k P^h_{ji} - \overset{c}{\nabla}_j P^h_{ki} = \\ = \frac{1}{n-1}\left(\delta^h_k(\overset{c}{\nabla}_\alpha P^\alpha_{ji} - \overset{c}{\nabla}_j P^\alpha_{\alpha i}) - \delta^h_j(\overset{c}{\nabla}_\alpha P^\alpha_{ki} - \overset{c}{\nabla}_k P^\alpha_{\alpha i})\right) - \\ - \frac{1}{n+1}(\delta^h_i(\overset{c}{\nabla}_j P^\alpha_{\alpha k} - \overset{c}{\nabla}_k P^\alpha_{\alpha j}) - \\ - \frac{1}{n-1}(\delta^h_k(\overset{c}{\nabla}_j P^\alpha_{\alpha i} - \overset{c}{\nabla}_i P^\alpha_{\alpha j}) - \delta^h_j(\overset{c}{\nabla}_k P^\alpha_{\alpha i} - \overset{c}{\nabla}_i P^\alpha_{\alpha k})))\end{gathered} \tag{12}$$

and for equiaffine spaces

$$\begin{gathered}\overset{c}{\nabla}_k P^h_{ji} - \overset{c}{\nabla}_j P^h_{ki} = \\ = \frac{1}{n-1}\left(\delta^h_k(\overset{c}{\nabla}_\alpha P^\alpha_{ji} - \overset{c}{\nabla}_j P^\alpha_{\alpha i}) - \delta^h_j(\overset{c}{\nabla}_\alpha P^\alpha_{ki} - \overset{c}{\nabla}_k P^\alpha_{\alpha i})\right).\end{gathered} \tag{13}$$

For further investigation the methods developed by [7] and [8] can be applied.

3 Proofs for theorems

Theorem 1. Proof

In a case of tensor field S of type $\binom{p}{q}$ covariant derivative of connection A_n (it will be designated by ∇) is defined in any coordinate system $x^1, x^2, \dots, x^n$ as follows:

$$\begin{gathered}\nabla_k S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) = \partial_k S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) + \Gamma^{i_1}_{k\alpha}(x) S^{\alpha i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) + \dots + \\ + \Gamma^{i_p}_{k\alpha}(x) S^{i_1 i_2 \dots i_{p-1}\alpha}_{j_1 j_2 \dots j_q}(x) - \Gamma^{\beta}_{kj_1}(x) S^{i_1 i_2 \dots i_p}_{\beta j_2 \dots j_q}(x) - \dots - \\ - \Gamma^{\beta}_{kj_q}(x) S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_{q-1}\beta}(x), \\ (i_1, \dots, i_p; \quad j_1, \dots j_q; \quad k = 1, 2, \dots, n).\end{gathered} \tag{14}$$

For the space $\bar{A}_n$ and covariant derivative $\bar{\nabla}$ in it the equation is true in a common coordinate system :

$$\begin{gathered}\bar{\nabla}_k S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) = \partial_k S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) + \bar{\Gamma}^{i_1}_{k\alpha}(x) S^{\alpha i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) + \dots + \\ + \bar{\Gamma}^{i_p}_{k\alpha}(x) S^{i_1 i_2 \dots i_{p-1}\alpha}_{j_1 j_2 \dots j_q}(x) - \bar{\Gamma}^{\beta}_{kj_1}(x) S^{i_1 i_2 \dots i_p}_{\beta j_2 \dots j_q}(x) - \dots - \\ - \bar{\Gamma}^{\beta}_{kj_q}(x) S^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_{q-1}\beta}(x), \\ (i_1, \dots, i_p; \quad j_1, \dots j_q; \quad k = 1, 2, \dots, n).\end{gathered} \tag{15}$$

Subtracting from the latter (14) and taking into account (2) we will get

$$\begin{gathered}\bar{\nabla}_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) - \nabla_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) = P^{i_1}_{k\alpha}(x) S^{\alpha i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \ldots + \\ + P^{i_p}_{k\alpha}(x) S^{i_1 i_2 \ldots i_{p-1}\alpha}_{j_1 j_2 \ldots j_q}(x) - P^{\beta}_{kj_1}(x) S^{i_1 i_2 \ldots i_p}_{\beta j_2 \ldots j_q}(x) - \ldots - \\ - P^{\beta}_{kj_q}(x) S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1}\beta}(x), \\ (i_1, \ldots, i_p; \quad j_1, \ldots j_q; \quad k = 1, 2, \ldots, n).\end{gathered} \tag{16}$$

The latter is true for any tensor and for deformation tensor (16). It takes a shape as follows:

$$\bar{\nabla}_k P^h_{ij}(x) - \nabla_k P^h_{ij}(x) = P^h_{k\alpha}(x) P^{\alpha}_{ij}(x) - P^{\alpha}_{ki}(x) P^h_{\alpha j}(x) - P^{\alpha}_{kj}(x) P^h_{i\alpha}(x). \tag{17}$$

Symmetrizing the latter we will get

$$\bar{\nabla}_k P^h_{ij} + \bar{\nabla}_j P^h_{ik} - \nabla_k P^h_{ij} - \nabla_j P^h_{ik} = -2 P^{\alpha}_{kj} P^h_{i\alpha}. \tag{18}$$

Alternating —

$$\bar{\nabla}_k P^h_{ij} - \bar{\nabla}_j P^h_{ik} - \nabla_k P^h_{ij} + \nabla_j P^h_{ik} = -2(P^h_{k\alpha} P^{\alpha}_{ij} - P^{\alpha}_{ki} P^h_{\alpha j}). \tag{19}$$

The law of change of curvature tensor that is defined as:

$$R^h_{\cdot\, ijk} = \partial_j \Gamma^h_{ik} + \Gamma^{\alpha}_{ik} \Gamma^h_{j\alpha} - \partial_k \Gamma^h_{ij} - \Gamma^{\alpha}_{ij} \Gamma^h_{k\alpha}, \tag{20}$$

in the course of the mapping of space A_n onto $\bar{A}_n$ can be written as follows:

$$\bar{R}^h_{\cdot\, ijk} = R^h_{\cdot\, ijk} + \nabla_k P^h_{ji} - \nabla_j P^h_{ki} + P^h_{\alpha k} P^{\alpha}_{ji} - P^h_{\alpha j} P^{\alpha}_{ki} \tag{21}$$

or taking into account (19),

$$\bar{R}^h_{\cdot\, ijk} = R^h_{\cdot\, ijk} + \frac{1}{2}(\nabla_k P^h_{ji} - \nabla_j P^h_{ki} + \bar{\nabla}_k P^h_{ji} - \bar{\nabla}_j P^h_{ki}). \tag{22}$$

Thus, the theorem is proved.

Theorem 2. Proof

From equation (5), taking into account (2), we will obtain

$$\overset{\lambda}{\Gamma}{}^h_{ij} = \frac{\Gamma^h_{ij} + \lambda \bar{\Gamma}^h_{ij}}{1+\lambda}. \tag{23}$$

For the deformation tensor the latter permits to write down:

$$\begin{gathered}\overset{\lambda}{\nabla}_k P^h_{ij}(x) - \nabla_k P^h_{ij}(x) = \\ = \frac{\lambda^2}{(1+\lambda)^2} \left(P^h_{k\alpha}(x) P^{\alpha}_{ij}(x) - P^{\alpha}_{ki}(x) P^h_{\alpha j}(x) - P^{\alpha}_{kj}(x) P^h_{i\alpha}(x)\right)\end{gathered} \tag{24}$$

and for Riemannian tensor — (6).

So the statement is proved.

Theorem 3. Proof

The reliability of theorem can easily be checked applying the definition of a medium connection and previous theorems.

Theorem 4. Proof

Weyl tensor W^h_{ijk} is defined as follows

$$W^h_{ijk} \stackrel{def}{=} R^h_{ijk} - \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right) + \\ + \frac{1}{n+1}\left(\delta^h_i R_{[jk]} - \frac{1}{n-1}\left(\delta^h_k R_{[ji]} - \delta^h_j R_{[ki]}\right)\right). \tag{25}$$

For equiaffine spaces the latter equation looks as follows

$$W^h_{ijk} \stackrel{def}{=} R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik}). \tag{26}$$

In respect to the mapping Weyl tensors A_n and $\bar{A}_n$ are connected by

$$\bar{W}^h_{ijk} = W^h_{ijk} + \overset{c}{\nabla}_k P^h_{ji} - \overset{c}{\nabla}_j P^h_{ki} - \\ - \frac{1}{n-1}\left(\delta^h_k(\overset{c}{\nabla}_\alpha P^\alpha_{ji} - \overset{c}{\nabla}_j P^\alpha_{\alpha i}) - \delta^h_j(\overset{c}{\nabla}_\alpha P^\alpha_{ki} - \overset{c}{\nabla}_k P^\alpha_{\alpha i})\right) + \\ + \frac{1}{n+1}(\delta^h_i(\overset{c}{\nabla}_j P^\alpha_{\alpha k} - \overset{c}{\nabla}_k P^\alpha_{\alpha j}) - \\ - \frac{1}{n-1}(\delta^h_k(\overset{c}{\nabla}_j P^\alpha_{\alpha i} - \overset{c}{\nabla}_i P^\alpha_{\alpha j}) - \delta^h_j(\overset{c}{\nabla}_k P^\alpha_{\alpha i} - \overset{c}{\nabla}_i P^\alpha_{\alpha k}))), \tag{27}$$

or for equiaffine spaces

$$\bar{W}^h_{ijk} = W^h_{ijk} + \overset{c}{\nabla}_k P^h_{ji} - \overset{c}{\nabla}_j P^h_{ki} - \\ - \frac{1}{n-1}\left(\delta^h_k(\overset{c}{\nabla}_\alpha P^\alpha_{ji} - \overset{c}{\nabla}_j P^\alpha_{\alpha i}) - \delta^h_j(\overset{c}{\nabla}_\alpha P^\alpha_{ki} - \overset{c}{\nabla}_k P^\alpha_{\alpha i})\right). \tag{28}$$

Thus, the theorem is proved.

References

[1] BOLSINOV A., KIOSAK, V. A., MATVEEV, V. S.: *Fubini Theorem for pseudo-Riemannian metric*,Journal of the London Mathematical Society., v. 80, 2, 2009, pp. 341 - 356.

[2] KIOSAK, V., HINTERLEITNER, I.:*φ(Ric)–Vector Fields in Riemannian Spaces*, Archivum-mathematicum, Brno, v. 44, 2008, pp. 385 - 390.

[3] KIOSAK, V. A., MATVEEV, V. S.: *Proof of the Projective Lichnerowicz Conjecture for Pseudo-Riemannian Metrics with Degree of Mobility Greater than Two*, Communications in Mathematical Physics, Springer, v.297, 2010, pp. 401 - 426.

[4] MATVEEV, V. S., KIOSAK, V. A., FEDOROVA, A., ROSEMANN S.: *The only closed Kahler manifold with degree of mobility >2 is (CP(n), g-Fubini-Study*, Proc. London Math. Soc., V.105, 1, 2012, pp. 153 - 188.

[5] MIKES, J., KIOSAK, V., VANZUROVA, A.:*Geodesic mappings of manifolds with affine connecti*, Palacky University, Olomouc, 2008, 220 p.

[6] NORDEN, A. P.: *Spaces of affine connection*, Nauka, Moskow, 1976, 432 p.

[7] RODIONOV, E. D., SLAVSKI, V. V., CHIBRIKOVA, L. N.:*Locally conformally homogeneous pseudo-Riemannian spaces*, Siberian Advances in Mathematics, v.17, 3, 2007, pp. 186 - 212.

[8] YANO, K.:*The theory of Lie derivatives and its applications*, North-Holland publishing Co., Amsterdam, Groningen, New-York, 1957.

Current address

Kiosak Volodymyr, doc. RNDr., CSc.
Institute of Engineering
Odessa State Academy of Civil Engineering and Architecture
Didrihsona st., 4, Odessa, 65029, Ukraine
E-mail: kiosakv@ukr.net

Lesechko Olexandr, doc. RNDr., CSc.
Institute of Engineering
Odessa State Academy of Civil Engineering and Architecture
Didrihsona st., 4, Odessa, 65029, Ukraine
E-mail: a.lesechko@ukr.net

Savchenko Olexandr, prof. RNDr., DrSc.
Economic Departament
Kherson State Agricultural University
Sretenskaya st., 23, Kherson, 73006, Ukraine
E-mail: savchenko.o.g@ukr.net