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On the Conformal Mappings of Special Quasi-Einstein Spaces

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Abstract. We have studied the conformal mappings of special quasi-Einstein spaces. When pseudo-Riemannian space V_n permits concircular mapping onto the quasi-Einstein space of the first type, then this space is an Einstein space. There is no quasi-Einstein space of the first type that differs from Einstein spaces permitting concircular mappings.

INTRODUCTION

A pseudo-Riemannian space V_n with a metric tensor g_{ij} is called an Einstein space, if the following conditions hold for the given space:

$$E_{ij} = 0, \tag{1}$$

where E_{ij} — the Einstein tensor, $E_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, R_{ij} — the Ricci tensor, R — the scalar curvature of the space V_n [2, 4].

Einstein spaces can be generalized in several ways. In particular, there are spaces having the Einstein tensor that differs from zero by a value pre-defined in some way

$$E_{ij} - D_{ij} = 0, \tag{2}$$

here D_{ij} — the certain tensor, called further the defect of Einstein tensor [1, 8, 9].

The limitations imposed on the defect of Einstein tensor can be either algebraic or differential depending on considerations grounded in physic reality.

The name of "quasi-Einstein" space was applied for the first time in order to denote the spaces, where:

$$D_{ij} = R_{\alpha i} R^{\alpha}_{j} - R_{\alpha\beta} R^{\alpha\ \ \beta}_{.ij.}, \tag{3}$$

where $R^i_j = g^{\alpha i} R_{\alpha j}$, $R^{h\ \ k}_{\ ij} = R_{\alpha ij\beta} g^{\alpha h} g^{\beta k}$, R_{ijkl} — Riemannian tensor V_n, g^{ij} — elements of reverse matrix of metrics tensor g_{ij}.

We shall say that these spaces are quasi-Einstein spaces of the first type.

This work aims at the study of conformal mappings of quasi-Einstein spaces of the first type.

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CONFORMAL MAPPINGS ONTO QUASI-EINSTEIN SPACES OF THE FIRST TYPE

Let V_n $(n > 2)$ be the pseudo-Riemannian space with metric tensor $g_{ij}(x)$ and $\bar{V}_n$ be another pseudo-Riemannian space with metric tensor $\bar{g}_{ij}(x)$. A conformal mapping is a bijection between points of two spaces V_n and $\bar{V}_n$ that

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \tag{4}$$

where σ — a function.

If σ is a constant, then the mapping is homothetic. Further, if it will not be specified, we will treat non-homothetic mappings. Objects of the space $\bar{V}_n$ that is conformly correspondent to the space V_n will be designated by a bar.

The (4) implies the following:

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij}.$$

The following formulae are true for Christoffel symbols:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}; \tag{5}$$

For a Riemannian tensor

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{h\alpha}(\sigma_{\alpha k} g_{ij} - \sigma_{\alpha j} g_{ik}) + \Delta_1\sigma(\delta^h_k g_{ij} - \delta^h_j g_{ik}); \tag{6}$$

For a Ricci tensor

$$\bar{R}_{ij} = R_{ij} + (n-2)\sigma_{ij} + (\Delta_2\sigma + (n-2)\Delta_1\sigma)\, g_{ij}; \tag{7}$$

For a scalar curvature

$$\bar{R} = e^{-2\sigma}\left(R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma\right). \tag{8}$$

Here and further $\quad \sigma_i \equiv \dfrac{\partial\sigma}{\partial x^i} \equiv \sigma_{,i}, \qquad \sigma^h = \sigma_\alpha g^{\alpha h},$

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}, \tag{9}$$

$\Delta_1\sigma$ and $\Delta_2\sigma$ are the first and second symbols of Beltrami defined in the following way:

$$\Delta_1\sigma = g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}; \qquad \Delta_2\sigma = g^{\alpha\beta}\sigma_{,\alpha\beta},$$

comma "," is a sign of covariant derivative by the connection V_n [4, 12, 13].

Let us treat conformal mappings of quasi-Einstein spaces of the first type.

Taking into account the equation (2) and (3), we will denote

$$A_{ij} = E_{ij} - R_{\alpha i}R^\alpha_j + R_{\alpha\beta}R^{\alpha\ \ \beta}_{\ ij}. \tag{10}$$

A change of object with respect to the given mapping is called a deformation.

Let us prove

Theorem 1. *If the pseudo-Riemannian space V_n is mapped conformally onto the pseudo-Riemannian space $\bar{V}_n$ then a deformation of tensor A_{ij} meets a condition*

$$\bar{A}_{ij} - A_{ij} = \overset{1}{\tau} D_{ij} + \overset{2}{\tau}\sigma_{\alpha\beta}R^{\alpha\ \ \beta}_{\ ij} + \overset{3}{\tau}(\sigma_{\alpha i}R^\alpha_j + \sigma_{\alpha j}R^\alpha_i) + \overset{4}{\tau}\sigma_{\alpha i}\sigma^\alpha_j + \overset{5}{\tau}\sigma_{ij} + \overset{6}{\tau}R_{ij} + \overset{7}{\tau}g_{ij}, \tag{11}$$

where $\overset{i}{\tau}$ $(i = 1, 2, \ldots, 7)$ — invariants.

Proof.

Let us calculate the change of the components of these equations in the course of conformal mappings. For Einstein tensor we will obtain [3]:

$$\bar{E}_{ij} = E_{ij} + (n-2)\left(\sigma_{ij} + \frac{1}{n}(\Delta_1\sigma - \Delta_2\sigma)g_{ij}\right). \tag{12}$$

And for tensors $R_{\alpha i}R^{\alpha}_{j}$ and $R_{\alpha\beta}R^{\alpha\ \beta}_{\ ij}$ respectively: taking into account (7)

$$\begin{aligned}\bar{R}_{\alpha i}\bar{R}^{\alpha}_{j} =& e^{-2\sigma}(R_{\alpha i}R^{\alpha}_{j} + (n-2)(\sigma_{\alpha i}R^{\alpha}_{j} + \sigma_{\alpha j}R^{\alpha}_{i}) \\ &+2(\Delta_2\sigma + (n-2)\Delta_1\sigma)(R_{ij} + (n-2)\sigma_{ij}) + (n-2)^2\sigma_{\alpha i}\sigma^{\alpha}_{j} + (\Delta_2\sigma + (n-2)\Delta_1\sigma)^2 g_{ij});\end{aligned} \tag{13}$$

and with respect to equations (6), (7)

$$\begin{aligned}\bar{R}_{\alpha\beta}\bar{R}^{\alpha\ \beta}_{\ ij} =& e^{-2\sigma}(R_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} + (n-2)\sigma_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} - 2(n-2)\sigma_{\alpha i}\sigma^{\alpha}_{j} \\ &-R_{\alpha j}\sigma^{\alpha}_{i} - R^{\alpha}_{i}\sigma_{\alpha j} + R_{ij}(\Delta_2\sigma + (n-2)\Delta_1\sigma - \Delta_1\sigma) + \sigma_{ij}(R + (n-2)(2\Delta_2\sigma + (n-4)\Delta_1\sigma) \\ &+g_{ij}(R_{\alpha\beta}\sigma^{\alpha\beta} + (n-2)\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \Delta_1\sigma R + (\Delta_2\sigma + (n-2)\Delta_1\sigma)^2 + (n-2)(\Delta_2\sigma - \Delta_1\sigma))).\end{aligned} \tag{14}$$

Taking into account (10), (12), (13) and (14), we obtain

$$\begin{aligned}\bar{A}_{ij} =& E_{ij} + (n-2)\left(\sigma_{ij} + \frac{1}{n}(\Delta_1\sigma - \Delta_2\sigma)g_{ij}\right) - e^{-2\sigma}(R_{\alpha i}R^{\alpha}_{j} + (n-2)(\sigma_{\alpha i}R^{\alpha}_{j} + \sigma_{\alpha j}R^{\alpha}_{i}) \\ &+2(\Delta_2\sigma + (n-2)\Delta_1\sigma)(R_{ij} + (n-2)\sigma_{ij}) + (n-2)^2\sigma_{\alpha i}\sigma^{\alpha}_{j} + (\Delta_2\sigma + (n-2)\Delta_1\sigma)^2 g_{ij}) \\ &+e^{-2\sigma}(R_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} + (n-2)\sigma_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} - 2(n-2)\sigma_{\alpha i}\sigma^{\alpha}_{j} - R_{\alpha j}\sigma^{\alpha}_{i} - R^{\alpha}_{i}\sigma_{\alpha j} + R_{ij}(\Delta_2\sigma + (n-2)\Delta_1\sigma - \Delta_1\sigma) \\ &+\sigma_{ij}(R + (n-2)(2\Delta_2\sigma + (n-4)\Delta_1\sigma) + g_{ij}(R_{\alpha\beta}\sigma^{\alpha\beta} + (n-2)\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \Delta_1\sigma R \\ &+(\Delta_2\sigma + (n-2)\Delta_1\sigma)^2 + (n-2)(\Delta_2\sigma - \Delta_1\sigma)).\end{aligned}$$

Collecting the similar terms we get

$$\begin{aligned}&\bar{A}_{ij} = A_{ij} - (e^{-2\sigma} - 1)(R_{\alpha i}R^{\alpha}_{j} - R_{\alpha\beta}R^{\alpha\ \beta}_{\ ij}) \\ &+ e^{-2\sigma}[(n-2)\sigma_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} - (n-1)(\sigma_{\alpha i}R^{\alpha}_{j} + \sigma_{\alpha j}R^{\alpha}_{i})n(n-2)\sigma_{\alpha i}\sigma^{\alpha}_{j} \\ &+ (R + (n-2)(e^{2\sigma} - n\Delta_1\sigma))\sigma_{ij} - (\Delta_2\sigma + (n-1)\Delta_1\sigma)R_{ij} \\ &+ g_{ij}(R_{\alpha\beta}\sigma^{\alpha\beta} + (n-2)\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \Delta_1\sigma R + (\Delta_2\sigma\Delta_1\sigma)(n-2)\left(1 - \frac{e^{2\sigma}}{n}\right))].\end{aligned} \tag{15}$$

Thus, the theorem is proved.

The proved theorem has the following corollaries:

Corollary 1. *If pseudo-Riemannian space V_n admits a conformal mapping with preservation of tensor A_{ij}, then the tensor complies with the condition*

$$\begin{aligned}&(1 - e^{-2\sigma})A_{ij} = e^{-2\sigma}[(n-2)\sigma_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} - (n-1)(\sigma_{\alpha i}R^{\alpha}_{j} + \sigma_{\alpha j}R^{\alpha}_{i}) \\ &- n(n-2)\sigma_{\alpha i}\sigma^{\alpha}_{j} + (R + (n-2)(e^{2\sigma} - n\Delta_1\sigma))\sigma_{ij} - (\Delta_2\sigma + (n-1)\Delta_1\sigma + e^{2\sigma} - 1)R_{ij} \\ &+ g_{ij}\left(R_{\alpha\beta}\sigma^{\alpha\beta} + (n-2)\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \Delta_1\sigma R + (\Delta_2\sigma - \Delta_1\sigma)(n-2)\left(1 - \frac{e^{2\sigma}}{n}\right) + (1 - e^{2\sigma})\frac{R}{n}\right)].\end{aligned} \tag{16}$$

Obviously, if $\bar{A}_{ij} = A_{ij}$, then (15) taking into account (10) we get (16).

Corollary 2. *If pseudo-Riemannian space V_n admits a conformal mapping onto quasi-Einstein space of the first type, then tensor A_{ij} meets the condition*

$$\begin{aligned}A_{ij} =& (n-1)(\sigma_{\alpha i}R^{\alpha}_{j} + \sigma_{\alpha j}R^{\alpha}_{i}) - (n-2)\sigma_{\alpha\beta}R^{\alpha\ \beta}_{\ ij} \\ &+n(n-2)\sigma_{\alpha i}\sigma^{\alpha}_{j} - (R + (n-2)(e^{2\sigma} - n\Delta_1\sigma))\sigma_{ij} + (\Delta_2\sigma + (n-1)\Delta_1\sigma + e^{2\sigma} - 1)R_{ij} \\ &-g_{ij}\left(R_{\alpha\beta}\sigma^{\alpha\beta} + (n-2)\sigma_{\alpha\beta}\sigma^{\alpha\beta} + \Delta_1\sigma + (\Delta_2\sigma - \Delta_1\sigma)(n-2)\left(1 - \frac{e^{2\sigma}}{n}\right) + (1 - e^{2\sigma})\frac{R}{n}\right).\end{aligned} \tag{17}$$

EQUIDISTANT QUASI-EINSTEIN SPACES

A pseudo-Riemannian space V_n having a metric tensor g_{ij} is called an equidistant, if it contains a vector field $\varphi_i \neq 0$ complying with equations

$$\varphi_{i,j} = \tau g_{ij}, \tag{18}$$

where τ — an invariant. When $\tau \neq 0$ it is an equidistant space of main variety, while if $\tau = 0$ it is a space of peculiar variety [11, 14].

Vector field satisfying the conditions (18), was called a concircular by K. Yano.

Integrability conditions for the main equations (18) can be formulated in the following way

$$\varphi_\alpha R^\alpha_{ijk} = g_{ij}\tau_{,k} - g_{ik}\tau_{,j}. \tag{19}$$

We get from the latter:

$$\tau_{,i} = \frac{1}{n-1}\varphi_\alpha R^\alpha_i. \tag{20}$$

The set of equations (18) and (20) is closed. It is a system of linear differential equations in covariant derivatives of the first order of Cauchy type with coefficients unequivocally defined by the space V_n, in respect to an unknown vector φ_i and invariant τ.

Let us note that equidistant spaces play crucial role in the theory of geodesic mappings. Or in other words they are extremely important for the theory of modeling with preservation of geodesic lines [3, 7].

The integrability conditions (18) are the basis for an evident conclusion:

$$\tau_{,k} = B\varphi_k, \tag{21}$$

here B is an invariant.

Then equations (19) and (20) can be re-written as:

$$\varphi_\alpha R^\alpha_{ijk} = B(\varphi_k g_{ij} - \varphi_j g_{ik}), \tag{22}$$

$$\varphi_\alpha R^\alpha_i = (n-1)B\varphi_i. \tag{23}$$

Multiplying (2) by φ^i and wrapping by i , taking into account (3), (22), (23), we obtain for quasi-Einstein spaces of the first type

$$(n-1)B - \frac{R}{n} = B^2(n-1)n - BR. \tag{24}$$

Thus, one of the following is true: either

$$B = \frac{R}{n(n-1)}, \tag{25}$$

or

$$B = \frac{1}{n}. \tag{26}$$

So the theorem is proved:

Theorem 2. *Integrability conditions for equations (18) for quasi-Einstein spaces of the first type are as follows*

$$\varphi_\alpha Y^\alpha_{ijk} = 0 \tag{27}$$

or

$$\varphi_\alpha Z^\alpha_{ijk} = 0, \tag{28}$$

here

$$Y^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right), \tag{29}$$

and

$$X^h_{ijk} = R^h_{ijk} - \frac{1}{n}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right). \tag{30}$$

Tensor Y^h_{ijk} — is called a tensor of concircular curvature. Considering (23) and (30) we will get

$$X^h_{ijk} = Y^h_{ijk} - \frac{R-(n-1)}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right). \tag{31}$$

Taking into account the fact that invariant B is uniquely defined for a given pseudo-Riemannian space, we can divide the equidistant almost For the type A — (27) is true, while for the type B — (28) is true.

SPECIAL CONFORMAL MAPPINGS OF QUASI-EINSTEIN SPACES

Let us treat a tensor of the following shape

$$Z^h_{ijk} = R^h_{ijk} - B\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right), \tag{32}$$

then

$$Z^\alpha_{ij\alpha} = Z_{ij} = R_{ij} - B(n-1)g_{ij}. \tag{33}$$

The same tensor exists for any $\bar{V}_n$

$$\bar{Z}_{ij} = \bar{R}_{ij} - \bar{B}(n-1)\bar{g}_{ij}. \tag{34}$$

Subtracting the equation (34) from (33), we get

$$(\bar{R}_{ij} - R_{ij}) - (n-1)(\bar{B}\bar{g}_{ij} - Bg_{ij}) = \bar{Z}_{ij} - Z_{ij}.$$

Considering (4) and (7), we obtain

$$\sigma_{ij} - \rho g_{ij} = \frac{1}{n-2}(\bar{Z}_{ij} - Z_{ij}), \tag{35}$$

where

$$\rho = \frac{1}{n-2}\left((n-1)(\bar{B}e^{2\sigma} - B) - (\Delta_2\sigma + (n-2)\Delta_1\sigma)\right).$$

Thus, the following theorem is true:

Theorem 3. *If V_n and $\bar{V}_n$ are a pair of two conformly-correspondent pseudo-Riemannian spaces, then their tensors Z_{ij} and $\bar{Z}_{ij}$ meet the conditions (35).*

A conformal mapping of a pseudo-Riemannian space V_n onto $\bar{V}_n$, when

$$Z_{ij} = \bar{Z}_{ij}, \tag{36}$$

is called a conformal mapping which preserves a tensor Z_{ij}.

Selecting a special type of tensor Z_{ij}, one can obtain different pseudo-Riemannian spaces, that are characterized by some pre-selected features [6, 10].

Considering (36) we transform the equations (35) in the following way:

$$\sigma_{ij} = \rho g_{ij} \tag{37}$$

or

$$\sigma_{ij} = \frac{\Delta_2\sigma - \Delta_1\sigma}{n} g_{ij}. \tag{38}$$

A special type of conformal mappings is defined according to their ability to preserve geodesic circles.

Definition 1. *A curve in pseudo-Riemannian space V_n is called a geodesic circle, if the first curvature of the curve is constant and the second curvature identically equals to zero.*

Definition 2. *A conformal mapping of pseudo-Riemannian space V_n, that preserves the geodesic circles, namely every geodesic circle of the space V_n maps into a geodesic circle of the conformal space $\bar{V}_n$, is called a concircular.*

A conformal mapping is concircular, if and only if the function σ complies with equations (4), (9), conditions (38).

Taking the latter into consideration we formulate the following

Theorem 4. *If a conformal mapping of pseudo-Riemannian spaces V_n preserves tensor Z_{ij}, then it preserves the geodesic circles too.*

On the other hand, considering (37) and (38), it is clear that if V_n admits concircular mappings and meets the condition

$$\frac{2\Delta_2 + (n-2)\Delta_1}{n} = \bar{B}e^{2\sigma} - B, \tag{39}$$

then this mapping preserves also the tensor Z_{ij}.

EQUATIONS IN LINEAR FORM

In order to study conformal mapping of pre-selected pseudo-Riemannian spaces, the linear form of equations is applied. The latter permits more effective enquiry. Let us transform the obtained results into the linear form.

Let us consider invariant S that satisfies the equation:

$$\sigma = -\ln|S|. \tag{40}$$

Then (4) can be re-written as

$$\bar{g}_{ij}(x) = S^{-2}g_{ij}.$$

Subsequently differentiating (40), we obtain

$$\sigma_{,i} = -\frac{1}{S}S_{,i} \qquad \sigma_{,ij} = -(S \cdot S_{,ij} - S_{,i}S_{,j}) \cdot S^{-2} \qquad \sigma_{ij} = -S_{,ij} \cdot S^{-1}$$

and

$$\Delta_1\sigma = \Delta_1 S \cdot S^{-2}; \qquad \Delta_2\sigma = (\Delta_1 S - S\Delta_2 S) \cdot S^{-2}.$$

Taking it into account, equations (38) can be transformed in the following way

$$S_{,ij} = \frac{\Delta_2 S}{n} g_{ij}, \tag{41}$$

and (39) —

$$S\Delta_2 S = \frac{n}{2}\Delta_1 S. \tag{42}$$

Vector $S_{,i}$ is called a concircular, if it satisfies the conditions (41) and a space is called an equidistant if it admits such a field.

Thus, we proved

Theorem 5. *If pseudo-Riemannian space V_n admits a conformal mapping preserving a tensor Z_{ij}, then V_n is an equidistant space.*

If $\Delta_2 S \neq 0$, then the equidistant space belongs to the main type. If $\Delta_2 S \equiv 0$ then the equidistant space belongs to the special type. If vector $S_{,i}$ is isotropic, namely $\Delta_1 S = 0$, then the equidistant space belongs to special type with a necessity. Equidistant spaces of the main type contain a special coordinate system where a metric tensor of an equidistant space can be written as follows

$$ds_n^2 = dx^{12} + f(x_1)ds_{n-1}^2(x_2, ..., x_n). \tag{43}$$

Here $f(x^1) \neq 0$ is a function, and ds_{n-1}^2 is a metric of $(n-1)$ – dimensional pseudo-Riemannian space.

Taking Theorem 5. into account, we formulate

Theorem 6. *If a pseudo-Riemannian space $V_n (n > 2)$ admits a conformal mapping preserving a tensor Z_{ij} and $\Delta_2 S \neq 0$, then there exists a system of coordinates where its metric tensor is defined by the equation (43).*

Considering (21), (33) and (42) we obtain the following expression for a concircular mapping of a quasi-Einstein space of the first type

$$\frac{1}{n}(\Delta_2 S)_{,i} = BS_{,i}, \tag{44}$$

where $B = \frac{R}{n(n-1)}$ — for type A, or $B = \frac{1}{n}$ — for type B.

Considering the substitution by (40), formulae (15), (16), (17) can be re-written as follows

$$\begin{aligned}\bar{A}_{ij} &= A_{ij} - (S^2-1)(R_{\alpha i}R^{\alpha}_{j} - R_{\alpha\beta}R^{\alpha\ \beta}_{\ ij}) - S\left((n-2)S_{,\alpha\beta}R^{\alpha\ \beta}_{\ ij} - (n-1)(S_{,\alpha i}R^{\alpha}_{j} + S_{,\alpha j}R^{\alpha}_{i})\right)\\ &- n(n-2)S_{,\alpha i}S^{\alpha}_{j} - S(R + (n-2)(S^{-2} - n\Delta_1 S^{-2}))S_{,ij} - (n\Delta_1 S - S\Delta_2 S)R_{ij}\\ &+ g_{ij}\left(-SR_{\alpha\beta}S^{\alpha\beta}_{,} + (n-2)S_{,\alpha\beta}S^{\alpha\beta}_{,} + R\Delta_1 S - (n-2)S\Delta_2 S\left(1 - \frac{1}{nS^2}\right)\right);\end{aligned} \tag{45}$$

$$\begin{aligned}A_{ij} &= \frac{S}{1-S^2}\left(-S_{,\alpha\beta}R^{\alpha\ \beta}_{\ ij} + (n-1)(S_{,\alpha i}R^{\alpha}_{j} + S_{,\alpha j}R^{\alpha}_{i}) - (R + (n-2)\frac{S^2}{1-n\Delta_1 S})\right)S_{,ij}\\ &- \frac{n(n-2)}{1-S^2}S_{,\alpha i}S^{\alpha}_{j} - \frac{1}{1-S^2}(n\Delta_1 S - S\Delta_2 S + 1 - S^2)R_{ij}\\ &+ \frac{1}{1-S^2}g_{ij}\left(-SS^{\alpha\beta}_{,}R_{\alpha\beta} + (n-2)S_{,\alpha\beta}S^{\alpha\beta}_{,} + R\Delta_1 S + \frac{(S^2-1)R}{n} - \frac{\Delta_2 S}{S}\frac{(n-2)}{n}(nS^2-1)\right);\end{aligned} \tag{46}$$

$$\begin{aligned}A_{ij} &= (n-2)\frac{1}{S}S_{,\alpha\beta}R^{\alpha\ \beta}_{\ ij} - (n-1)\frac{1}{S}(S_{,\alpha i}R^{\alpha}_{j} + S_{,\alpha j}R^{\alpha}_{i}) + n(n-2)\frac{1}{S^2}S_{,\alpha i}S^{\alpha}_{j}\\ &+ (R + (n-2)(S^{-2} - S^{-2}\Delta_1 S))S_{,ij}\frac{1}{S} - \frac{1}{S^2}(n\Delta_1 S - S\Delta_2 S - S^2 + 1)R_{ij}\\ &+ \frac{1}{S^2}g_{ij}\left(-SS^{\alpha\beta}_{,}R_{\alpha\beta} + (n-2)S_{,\alpha\beta}S^{\alpha\beta}_{,} + R\Delta_1 S + \frac{(S^2-1)}{n}R - \frac{\Delta_2 S}{S}\frac{(n-2)}{n}(nS^2-1)\right).\end{aligned} \tag{47}$$

Here $S^{i}_{j} = g^{\alpha i}S_{,\alpha j}; S^{ij}_{,} = g^{\alpha i}g_{\beta j}S_{,\alpha\beta}$. Thus, in the case of conformal mapping of pseudo-Riemannian space V_n onto the pseudo-Riemannian space $\bar{V}_n$, tensors A_{ij} and$\bar{A}_{ij}$ are connected by conditions (45), where S — can be defined via the expression (42). In order to make the pseudo-Riemannian space V_n permitting the conformal mappings with preservation of tensor A_{ij}, the latter should conform to limitations (46). On the other hand, when pseudo-Riemannian space V_n permits conformal mappings on the quasi-Einstein space of the first type, then tensor A_{ij} conforms to the condition (47).

CONCIRCULAR MAPPINGS OF QUASI-EINSTEIN SPACES OF THE FIRST TYPE

Let us turn our attention to quasi-Einstein spaces of the first type permitting concircular mappings. The latter spaces are equidistant and thus, consist of two classes: A and B.

Einstein tensor and tensor of concircular curvature are invariant in the course of concircular mappings.

Let us formulate the theorem for tensor A_{ij}:

Theorem 7. *When pseudo-Riemannian space V_n permits concircular mapping onto the pseudo-Riemannian space $\bar{V}_n$, then their tensors A_{ij} and $\bar{A}_{ij}$ satisfy conditions*

$$\bar{A}_{ij} - A_{ij} = -(S^2-1)(R_{\alpha i}R^{\alpha}_{j} - R_{\alpha\beta}R^{\alpha\ \beta}_{\ ij}) + (2S\Delta_2 S - n\Delta_1 S)E_{ij} + \tau g_{ij}. \tag{48}$$

Here τ — an invariant.

In order to verify the theorem, let us substitute (41) into (45). Having been transformed the resulting expression will take a look of (47), where

$$\tau = -\frac{\Delta_2 S}{nS}\left((n^2-2)(S^2-1) - 2(n-1)\right).$$

When $\bar{A}_{ij} = 0$, or otherwise $\bar{V}_n$ quasi-Einstein space of the first type, then for V_n the following is true

$$(1 - S^2 + 2S\Delta_2 S - n\Delta_1 S)E_{ij} + \tau g_{ij} = 0. \tag{49}$$

Wrapping, we will see that $\tau = 0$, or in other words

$$\Delta_2 S = 0.$$

Then either $1 - S^2 - n\Delta_1 S = 0$, or $E_{ij} = 0$.

Let us treat the case, when

$$S_{,\alpha} S^{\alpha} = \frac{1 - S^2}{n}, \tag{50}$$

then

$$(\Delta_1 S)_{,i} = -\frac{2S}{n} S_{,i} = 0.$$

It is impossible, then

Corollary 3. *When pseudo-Riemannian space V_n permits concircular mapping onto the quasi-Einstein space of the first type, then this space is an Einstein space.*

And considering the invariance of Einstein tensor with respect to concircular mappings we get

Corollary 4. *There is no quasi-Einstein space of the first type that differs from Einstein spaces permitting concircular mappings.*

Let us note that Einstein spaces permitting the covariant constant vector are Ricci flat.

CONCLUSION

The constant search for a better model leads to the need of investigation of quasi-Einstein spaces of the first type. Conformal mappings, namely mappings preserving angles, are extremely important in the theory of modeling of pseudo-Riemannian spaces. The obtained results are the premise for further investigations on other types of mappings of quasi-Einstein spaces, in particular geodesic mappings.

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