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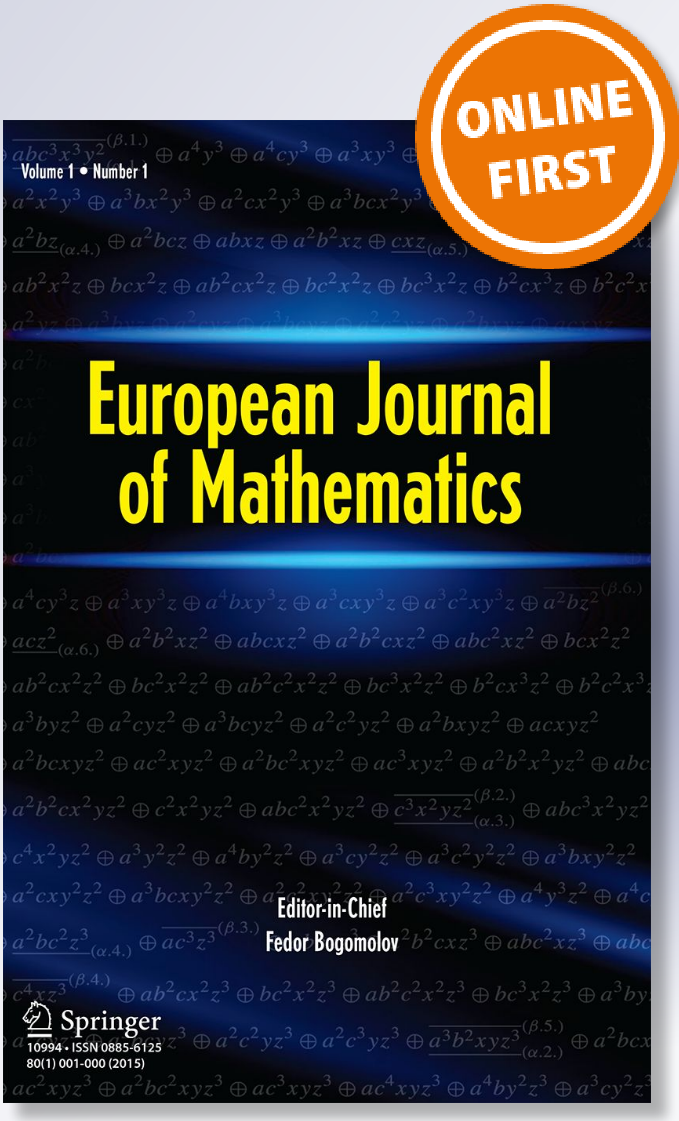

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RESEARCH ARTICLE

Fuzzy metrization of the spaces of idempotent measures

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Abstract
In idempotent mathematics, the idempotent measures (Maslov measures) are counterparts of the probability measures. We provide a fuzzy metrization of the set of idempotent measures on fuzzy metric spaces. We prove that this fuzzy metrization determines a monad in the category of fuzzy metric spaces and non-expanding maps.

Keywords Fuzzy metric · Idempotent measure · Hyperspace · Monad

Mathematics Subject Classification 54B20 · 54B30 · 46E27

1 Introduction

Fuzzy metric spaces are generalizations of probabilistic metric spaces defined by Menger [9]. They find numerous applications, e.g. to color image filtering [4,10]. Some results concerning fixed points of maps of these spaces can be applied to Baire spaces in the domains of words [13,14].

There are two main approaches to the definition of fuzzy metric spaces. We use the one introduced by George and Veeramaani [6]. The class of fuzzy metrizable spaces in the sense of George and Veeramaani coincides with that of metrizable spaces. This naturally suggests that various constructions of topology of metric spaces have their

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counterparts in the realm of fuzzy metric spaces. One of them is a fuzzy counterpart of the ultrametric, namely, the so-called fuzzy ultrametric. The fuzzy ultrametrization of spaces of probability measures with compact supports on fuzzy ultrametric spaces is constructed in [17].

The present note is devoted to such a fuzzy metrization. We use an approach based on the notion of density of an idempotent measure [1]. Note that idempotent measures are analogs of probability measures in idempotent mathematics, a part of mathematics in which at least one of the arithmetic operations on the reals is replaced by an idempotent operation (e.g. maximum). In [8], a fuzzy ultrametrization of the set of idempotent measures (Maslov measures) with compact support on fuzzy ultrametric spaces is constructed and this leads to a natural question of fuzzy metrization of the set of idempotent measures with compact support on a fuzzy metric space.

The main result of this paper states that there exists a fuzzy natural metric on the set of idempotent measures on a compact fuzzy metric space. Our construction is based on the fuzzy Hausdorff metrization of the hyperspaces of fuzzy metric spaces investigated in [12]. In some sense, the obtained metrization is closer to the Prokhorov metric rather than to the Kantorovich one on the spaces of probability measures. Moreover, we show that the functor of idempotent measure determines a monad in the category of fuzzy metric spaces and non-expanding maps.

2 Preliminaries

2.1 Idempotent measures

We recall necessary information on the spaces of idempotent measures; see, e.g. [19] for more details. Let X be a compact Hausdorff space. As usual, by $C(X)$ we denote the space of continuous functions on X endowed with the sup-norm. For any $c \in \mathbb{R}$, we denote by c_X the constant function on X taking the value c.

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. We consider the natural order on $\mathbb{R}_{\max}$. We also use the following traditional notation in idempotent mathematics: $\oplus$ for max and $\odot$ for $+$ (this may concern either numbers or functions). The following convention is used: $-\infty \odot x = x \odot -\infty = -\infty$.

A functional $\mu \colon C(X) \to \mathbb{R}$ is said to be an *idempotent measure* (Maslov measure) if it satisfies the following conditions:

- $\mu(c_X) = c$;
- $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$;
- $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$.

The Dirac measure δ_x concentrated at $x \in X$ (i.e. $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$) is an example of idempotent measure. A more complicated example is $\mu = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i}$, where $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in [-\infty, 0]$ are such that $\bigoplus_{i=1}^n \lambda_i = 0$. Here $\mu(\varphi) = \bigoplus_{i=1}^n \lambda_i \odot \varphi(x_i)$.

We denote by $I(X)$ the set of all idempotent measures on X. The set $I(X)$ is endowed with the weak* topology. A base of this topology consists of sets of the form

$$O\langle\mu_0;\varphi_1,\ldots,\varphi_k;\varepsilon\rangle=\left\{\mu\in I(X)\mid|\mu(\varphi_i)-\mu_0(\varphi_i)|<\varepsilon,\ i=1,\ldots,n\right\},$$

where $\mu_0\in I(X),\varphi_1,\ldots,\varphi_k\in C(X),\varepsilon>0$. The space $I(X)$ is a compact Hausdorff space if so is X (see [19]). If, in addition X is metrizable, then so is $I(X)$. The map $\eta=\eta_X\colon X\to I(X)$, $\eta(x)=\delta_x$, is an embedding.

Let $f\colon X\to Y$ be a continuous map of compact Hausdorff spaces. The map $I(f)\colon I(X)\to I(Y)$ is defined by the formula: $I(f)(\mu)(\varphi)=\mu(\varphi f)$, $\mu\in I(X)$, $\varphi\in C(Y)$. This map is continuous [19]. The construction I determines a functor in the category **Comp** of compact Hausdorff spaces and continuous maps.

2.2 Fuzzy metric spaces

We provide necessary information on the fuzzy metric spaces in the sense of George and Veeramani; see, e.g. [6,7] for more details.

Definition 2.1 A binary operation $*\colon[0,1]\times[0,1]\to[0,1]$ is called a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a*1=a$ for all $a\in[0,1]$,
(iv) $a*b\leqslant c*d$ whenever $a\leqslant c$ and $b\leqslant d$, for each $a,b,c,d\in[0,1]$.

The following are examples of continuous t-norms: 1) $a*b=ab$, 2) $a*b=\min\{a,b\}$, 3) $a*b=\max\{a+b-1,0\}$ (Łukasiewicz t-norm).

Definition 2.2 A 3-tuple $(X,M,*)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t\in(0,\infty)$:

(i) $M(x,y,t)>0$;
(ii) $M(x,y,t)=1$ if and only if $x=y$;
(iii) $M(x,y,t)=M(y,x,t)$;
(iv) $M(x,y,t)*M(y,z,s)\leqslant M(x,z,t+s)$;
(v) the function $M(x,y,\cdot)\colon(0,\infty)\to[0,1]$ is continuous.

If $(X,M,*)$ is a fuzzy metric space, $(M,*)$ is called a *fuzzy metric* on X.

Let X be a topological space. By $\exp X$ we denote the family of all nonempty compact subsets in X. The set $\exp X$ is endowed with the Vietoris topology; its base comprises of sets of the form

$$\langle U_1,\ldots,U_n\rangle=\left\{A\in\exp X\mid A\subset\bigcup_{i=1}^{n}U_i,\ A\cap U_i\neq\varnothing,\ i=1,\ldots,n\right\},$$

where $U_1,\ldots,U_n$, $n\in\mathbb{N}$, are open sets in X. Let $f\colon X\to Y$ be a continuous map of topological spaces. The map $\exp f\colon\exp X\to\exp Y$ defined by the formula $\exp f(A)=f(A)$, $A\in\exp X$, is continuous. The construction exp determines a functor in the category **Comp**.

Let $(X, M, *)$ be a fuzzy metric space. Given $a \in X$, $b \in \exp X$, and $t \in (0, \infty)$, define $M(a, B, t) = \sup\{M(a, b, t) \mid b \in B\}$. For every $A, B \in \exp X$ and $t > 0$, define $H_M \colon \exp X \times \exp X \times (0, \infty) \to [0, 1]$ by

$$H_M(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\right\}.$$

Then $(\exp X, H_M, *)$ is a fuzzy metric space. The function $(H_M, *)$ is called the *Hausdorff fuzzy metric* on the set $\exp X$. The Hausdorff fuzzy metric generates the Vietoris topology on $\exp X$. Let $(X_i, M_i, *)$, $i = 1, 2$, be fuzzy metric spaces.

Definition 2.3 A map $f \colon X_1 \to X_2$ is called an *isometric embedding* if, for every $(x, y, t) \in X \times X \times (0, \infty)$, $M_2(f(x), f(y), t) = M_1(x, y, t)$.

Definition 2.4 A map $f \colon X_1 \to X_2$ is called *non-expanding* if, for every $(x, y, t) \in X \times X \times (0, \infty)$, $M_2(f(x), f(y), t) \geqslant M_1(x, y, t)$.

The fuzzy metric spaces and their non-expanding maps form a category. The function

$$M \colon (X_1 \times X_2) \times (X_1 \times X_2) \times (0, \infty) \to [0, 1],$$

defined by the formula $M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t)$, is a fuzzy metric on $X_1 \times X_2$ (see, e.g. [11]). This metric generates the product topology on $X_1 \times X_2$.

3 Fuzzy metrics on the set of idempotent measures

Let X be a compact metrizable space. Let $\overline{I}(X)$ denote the set of $A \in \exp(X \times [-\infty, 0])$ satisfying the following conditions:

- $(x, -\infty) \in A$ for every $x \in X$;
- there exists $x \in X$ such that $(x, 0) \in A$;
- $(x, t) \in A$ implies $(x, s) \in A$, for every $s \in [-\infty, t]$.

Proposition 3.1 *The set* $\overline{I}(X)$ *is closed in* $\exp(X \times [-\infty, 0])$.

Proof We see that $\overline{I}(X) = K_1 \cap K_2 \cap K_3$, where

$$\begin{aligned} K_1 &= \left\{A \in \exp(X \times [-\infty, 0]) \mid A \supset X \times \{-\infty\}\right\}, \\ K_2 &= \left\{A \in \exp(X \times [-\infty, 0]) \mid A \cap (X \times \{0\}) \neq \varnothing\right\}, \\ K_3 &= \left\{A \in \exp(X \times [-\infty, 0]) \mid (x, t) \in A \text{ implies } \{x\} \times [-\infty, t] \subset A\right\}. \end{aligned}$$

From the definition of the Vietoris topology it easily follows that the sets K_1 and K_2 are closed in $\exp(X \times [-\infty, 0])$, therefore it suffices to prove that the set $\exp(X \times [-\infty, 0]) \setminus K_3$ is open in $\exp(X \times [-\infty, 0])$.

Let $A \in \exp(X \times [-\infty, 0]) \setminus K_3$, then there exist $x \in X$ and $t, t' \in [-\infty, 0]$ such that $(x, t) \in A$, $(x, t') \notin A$ and $-\infty \leqslant t' < t$. Since A is closed in $X \times [-\infty, 0]$, there exist neighborhoods U of x and V, W of t' and t in $[-\infty, 0]$ respectively such that $(\overline{U} \times \overline{V}) \cap A = \varnothing$, $W \cap V = \varnothing$, and $V \subset [0, t)$.

Then

$$A \in \langle X \times [0, \infty], (X \times [0, \infty]) \setminus (\overline{U} \times \overline{V}), U \times W \rangle \subset \exp(X \times [-\infty, 0]) \setminus K_3$$

and we are done. □

For any continuous map $f : X \to Y$, let

$$\overline{I}(f) = \exp(f \times 1_{[-\infty,0]}) | \overline{I}(X) : \overline{I}(X) \to \overline{I}(Y).$$

It is easy to verify that $\overline{I}$ is a functor in the category **Comp**. Given $A \in \overline{I}(X)$, define $h(A) : C(X) \to \mathbb{R}$ as follows:

$$h(A)(\varphi) = \bigoplus \{t \odot \varphi(x) \mid (x, t) \in A\}.$$

Proposition 3.2 *For every $A \in \overline{I}(X)$, $h(A) \in I(X)$.*

Proof Clearly, $h(A)(c_X) = c$. Given $\varphi, \psi \in C(X)$, we obtain

$$\begin{aligned} h(A)(\varphi \oplus \psi) &= \bigoplus \left\{t \odot (\varphi \oplus \psi)(x) \mid (x, t) \in A\right\} \\ &= \bigoplus \left\{((t \odot \varphi) \oplus (t \odot \psi))(x) \mid (x, t) \in A\right\} \\ &= \bigoplus \left\{(t \odot \varphi)(x) \oplus (t \odot \psi)(x) \mid (x, t) \in A\right\} \\ &= \bigoplus \left\{(t \odot \varphi)(x) \mid (x, t) \in A\right\} \oplus \bigoplus \left\{(t \odot \psi)(x) \mid (x, t) \in A\right\} \\ &= h(A)(\varphi) \oplus h(A)(\psi). \end{aligned}$$

For any $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} h(A)(\lambda \odot \varphi) &= \bigoplus \{t \odot (\lambda \odot \varphi)(x) \mid (x, t) \in A\} \\ &= \lambda \odot \bigoplus \{t \odot \varphi(x) \mid (x, t) \in A\} = \lambda \odot h(A)(\varphi). \end{aligned}$$ □

Proposition 3.3 *The map $h : \overline{I}(X) \to I(X)$ is continuous.*

Proof Let (A_i) be a sequence in $\overline{I}(X)$ converging to $A \in \overline{I}(X)$. We are going to show that $\lim_{i\to\infty} h(A_i) = h(A)$.

Let $\varphi \in C(X)$. We are going to demonstrate that $\lim_{i\to\infty} h(A_i)(\varphi) = h(A)(\varphi)$. Given $\varepsilon > 0$, find a finite open cover $\mathcal{U}$ of the space X such that the oscillation of φ on every element of $\mathcal{U}$ is less than $\varepsilon/2$. Let $c < 0$ be such that $c < \inf\{\varphi(x) \mid x \in X\}$. Consider a finite cover $\mathcal{V}$ of $[c, 0]$ such that the diameter of every element $\mathcal{V}$ is less than $\varepsilon/2$.

Consider the family $\mathcal{U}\times\mathcal{V}=\{U\times V\mid U\in\mathcal{U},\ V\in\mathcal{V}\}$. There is $i_0\in\mathbb{N}$ such that, for all $i\geqslant i_0$, the sets A_i and A are $(\mathcal{U}\times\mathcal{V})$-close. The latter means that

$$A\cap U\times V\neq\varnothing\quad\Longleftrightarrow\quad A_i\cap U\times V\neq\varnothing,$$

for all $U\in\mathcal{U}$, $V\in\mathcal{V}$.

There exists $(x,t)\in A$ such that $h(A)(\varphi)=t\odot\varphi(x)$. Then necessarily $(x,t)\in\bigcup(\mathcal{U}\times\mathcal{V})$, i.e. there exists $U\times V\in\mathcal{U}\times\mathcal{V}$ such that $(x,t)\in U\times V$. Since A_i and A are $(\mathcal{U}\times\mathcal{V})$-close, there is $(x',y')\in A_i\cap(U\times V)$. We see that

$$h(A_i)(\varphi)=t'\odot\varphi'(x)\geqslant h(A)(\varphi)=t\odot\varphi(x).$$

We conclude that $\lim_{i\to\infty}h(A_i)(\varphi)=h(A)(\varphi)$. Since φ is arbitrary, $\lim_{i\to\infty}h(A_i)=h(A)(\varphi)$, by the definition of the weak* topology. Thus, the map h is continuous. □

Proposition 3.4 *The map $h\colon\overline{I}(X)\to I(X)$ is onto.*

Proof Let $\mu=\bigoplus_{i=1}^n\lambda_i\odot\delta_{x_i}\in I(X)$. Consider

$$A=(X\times\{-\infty\})\cup\bigcup_{i=1}^n(\{x_i\}\times[-\infty,\lambda_i]).$$

Clearly, $A\in\overline{I}(X)$ and $h(A)=\mu$. Since the set of idempotent measures with finite support is dense in $I(X)$ (see [19]), we conclude that h is an onto map. □

Proposition 3.5 *The map $h\colon\overline{I}(X)\to I(X)$ is an embedding.*

Proof Suppose $A,B\in\overline{I}(X)$, $A\neq B$. Without loss of generality, one may assume that $A\setminus B\neq\varnothing$. Let $(x,t)\in A\setminus B$. One may also suppose that $(x,t')\notin A$ for all $t'>t$.

There exist a neighborhood U of x in X and $r<t$ such that $U\times(r,0]\cap B=\varnothing$. Let $\varphi\in C(X)$ be a function satisfying the following properties:

- $\varphi\leqslant 0$,
- $\varphi(x)=0$,
- $\varphi(y)\leqslant r$, for every $y\in X\setminus U$.

Then, clearly, $h(A)(\varphi)\geqslant\varphi(x)+t=t$ and $h(B)(\varphi)\leqslant r\neq t=h(A)(\varphi)$. Therefore $h(A)\neq h(B)$. □

Finally, we obtain that the map $h=h_X\colon\overline{I}(X)\to I(X)$ is a homeomorphism. Note also that $(h_X)\colon\overline{I}\to I$ is an isomorphism of functors $\overline{I}$ and I considered as endofunctors in the category of compact metrizable spaces and continuous maps.

Remark 3.6 The map $h_X\colon\overline{I}(X)\to I(X)$ can be defined for any compact Hausdorff spaces X, not only for metrizable. Using slightly more complicated arguments one can demonstrate that h_X is an isomorphism of $\overline{I}$ and I in **Comp**.

Let $*$ be the product t-norm. We endow the segment $[-\infty, 0]$ with the fuzzy metric N defined by the formula

$$N(x, y, t) = \frac{t}{t + |e^x - e^y|}$$

(naturally, $e^{-\infty} = 0$). Then, given a compact fuzzy metric space $(X, M, *)$, the space $X \times [-\infty, 0]$ is endowed with the fuzzy product metric and, consequently, the space $\overline{I}(X)$ is endowed with the fuzzy Hausdorff metric generated by this product metric. This allows us to endow the set $I(X)$ with the fuzzy metric induced from the mentioned metric by the map h.

Denote by F the functor of multiplication by $[-\infty, 0]$. For every X, let

$$m_X : F^2(X) = X \times [-\infty, 0] \times [-\infty, 0] \to F(X) = X \times [-\infty, 0]$$

denote the map acting by the formula $m_X(x, t, s) = (x, t \odot s)$.

Lemma 3.7 *The map m_X is non-expanding.*

Proof Denote the fuzzy metrics on X, $F^2(X)$, and $F(X)$ by M, L, and L' respectively. Let $(x, t, s), (x', t', s') \in F(X)$. Then $m_X(x, t, s) = (x, t \odot s)$, $m_X(x', t', s') = (x', t' \odot s')$. Given $r > 0$, we obtain

$$\begin{aligned}
&L((x, t, s), (x', t', s'), r) \\
&\quad = M(x, x', r) N(t, t', r) N(s, s', r) \\
&\quad = M(x, x', r) \cdot \frac{r}{r + |e^t - e^{t'}|} \cdot \frac{r}{r + |e^s - e^{s'}|} \\
&\quad = M(x, x', r) \cdot \frac{r^2}{r^2 + r(|e^t - e^{t'}| + |e^s - e^{s'}|) + |e^t - e^{t'}||e^s - e^{s'}|} \\
&\quad \leqslant M(x, x', r) \cdot \frac{r^2}{r^2 + r(|e^t - e^{t'}| + |e^s - e^{s'}|)} \\
&\quad \leqslant M(x, x', r) \cdot \frac{r}{r + |e^{t \odot s} - e^{t' \odot s'}|} \\
&\quad = L'(m_X(x, t, s), m_X(x', t', s')).
\end{aligned}$$

□

For any $x \in X$, denote by $\overline{\eta}(x) \in \overline{I}(X)$ the only point such that $h\overline{\eta}_X(x) = \eta_X(x) = \delta_x \in I(X)$.

Proposition 3.8 *The map $\overline{\eta}_X : X \to \overline{I}(X)$ is non-expanding.*

Proof Let $x_i \in X$, $i = 1, 2$. Then

$$\overline{\eta}_X(x_i) = (X \times \{-\infty\}) \cup (\{x_i\} \times [-\infty, 0]), \quad i = 1, 2.$$

Note that, for any $x \in X$, $M((x, -\infty), \overline{\eta}_X(x_2), r) = 1$. Suppose that $t \in [-\infty, 0]$. Then, for any $x \in X$, we obtain

$$M((x_1, t), (x, -\infty), r) = M(x_1, x, r)N(t, -\infty, r).$$

For every $s \in [-\infty, 0]$,

$$M((x_1, t), (x_2, s), r) = M(x_1, x_2, r)M(t, s, r) \leqslant M(x_1, x_2, r).$$

We conclude that, for any $(x, t) \in \eta_X(x_1)$,

$$\begin{aligned} M((x, t), \eta_X(x_2), r) &= \min\{1, \max\{M(x_1, x_2, r), N(t, -\infty, r)\}\} \\ &= \max\{M(x_1, x_2, r), N(t, -\infty, r)\} \geqslant M(x_1, x_2, r). \end{aligned}$$

Similarly, for any $(x, t) \in \overline{\eta}_X(x_2)$, $M((x, t), \overline{\eta}_X(x_1), r) \geqslant M(x_1, x_2, r)$. Summing up, we obtain $M_H(\overline{\eta}_X(x_1), \overline{\eta}_X(x_2), r) \geqslant M(x_1, x_2, r)$. □

The following example shows that the map $\overline{\eta}_X$, in general, is not an isometric embedding. Let $X = \{a, b\}$ be a metric space with $d(a, b) = 2$. This metric determines the standard fuzzy metric M_d on X by the formula $M_d(a, b, t) = t/(t + d(a, b)) = t/(t + 2)$. Denote by L the product fuzzy metric on $X \times [-\infty, 0]$ and by L_H the fuzzy metric on $\overline{I}(X)$ induced by the Hausdorff metric on $\exp(X \times [-\infty, 0])$. Then one can easily calculate that $L_H(\eta_X(a), \eta_X(b), t) \geqslant t/(t + 1)$ and therefore $L_H(\eta_X(a), \eta_X(b), t) \neq M_d(a, b, t)$.

Define a map $\lambda_X : F(\exp X) \to \exp F(X)$ as $\lambda_X(A, t) = \{(a, t) \mid a \in A\}$.

Lemma 3.9 *The map λ_X is non-expanding.*

Proof Let $(A, t), (B, s) \in F(\exp X)$, $r > 0$. Then

$$L((A, t), (B, s), r) = M_H(A, B, r)N(t, s, r).$$

Suppose that $M_H(A, B, r) \geqslant K$, for some $K \geqslant 0$. Then $L((A, t), (B, s), r) \geqslant K N(t, s, r)$. Let $(a, t) \in \lambda_X(A, t)$. By the definition of the fuzzy Hausdorff metric, there exists $b \in B$ such that $M(a, b, r) \geqslant K$ and therefore $M((a, t), (b, s), r) \geqslant K N(t, s, r)$. Similarly, for every $(b, s) \in \lambda_X(B, s)$, there exists $a \in A$ such that $M((a, t), (b, s), r) \geqslant K N(t, s, r)$. We conclude that $M_H(\lambda_X(A, t), \lambda_X(B, s)) \geqslant K N(t, s, r)$. □

Proposition 3.10 *Let $f : X \to Y$ be a non-expanding map of compact fuzzy metric spaces. Then the map $I(f) : I(X) \to I(Y)$ is also non-expanding.*

Proof Due to functorial isomorphism of the functors $\overline{I}$ and I, one can consider the map $\overline{I}(f) = \exp(f \times 1_{[-\infty,0]})|\overline{I}(X)$. One can easily verify that the map $f \times 1_{[-\infty,0]}$ is non-expanding. The statement of the proposition is then an immediate consequence of the fact that the hyperspace functor preserves the class of non-expanding maps (see [16]). □

Given a t-norm $*$, denote by $\mathcal{FMS}(*)$ the category whose elements are compact fuzzy metric spaces and whose morphisms are non-expanding maps of these spaces. Let $U\colon \mathcal{FMS}(*) \to \mathbf{Comp}$ denote the forgetful functor. The above results can be interpreted as follows: the functor I in the category **Comp** admits a lifting to the category $\mathcal{FMS}(*)$. Also, the maps λ_X form a natural transformation λ in the category **Comp**.

A monad on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \theta)$, where $T\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor, $\eta\colon 1_{\mathcal{C}} \to T$ and $\theta\colon T^2 \to T$ are natural transformations satisfying the conditions $\theta\eta_T = \theta T\eta = 1_T$ and $\theta T\theta = \theta\theta_T$ (see, e.g. [2] for details).

The hyperspace functor in the category **Comp** determines a monad structure. The obtained monad (the hyperspace monad; see [18] for its uniqueness) is denoted by $\mathbb{H} = (\exp, s, u)$. The natural transformation $s\colon 1_{\mathbf{Comp}} \to \exp$ is defined as $s_X(x) = \{x\}$, and the natural transformation $u\colon \exp^2 \to \exp$ is defined as $u_X(\mathcal{A}) = \bigcup \mathcal{A}$. Note that the same natural transformations allow us to define the hyperspace monad on the category $\mathcal{FMS}(*)$.

There exists a monad structure on the functor I acting in the category **Comp** (see [19]). Indeed, define the natural transformation $\eta\colon 1 \to I$ as follows: $\eta_X(x) = \delta_x$, $x \in X$. Given a function $\varphi \in C(X)$, let $\overline{\varphi}\colon I(X) \to \mathbb{R}$ be defined by the formula $\overline{\varphi}(\mu) = \mu(\varphi)$. Then $\overline{\varphi} \in C(X)$ and we define $\theta_X\colon I^2(X) \to I(X)$ by the formula $\theta_X(M)(\varphi) = M(\overline{\varphi})$, for every $M \in I^2(X)$ and $\varphi \in C(X)$.

Define $\overline{\theta}_X\colon \overline{I}^2(X) \to \overline{I}(X)$ by the formula

$$\overline{\theta}_X(\mathcal{A}) = \big\{(a, ts) \mid \text{ there is } (A, t) \in \mathcal{A} \text{ and there is } (a, s) \in A\big\}.$$

We are going to show that the diagram

$$\begin{array}{ccc} \overline{I}^2(X) & \xrightarrow{\;I(h_X)h_{\overline{I}(X)}\;} & I^2(X) \\ {\scriptstyle \overline{\theta}_X}\downarrow & & \downarrow{\scriptstyle \theta_X} \\ \overline{I}(X) & \xrightarrow[\;h_X\;]{} & I(X) \end{array} \tag{1}$$

is commutative. Let $\mathcal{A} \in \overline{I}^2(X)$. Then

$$h_X\overline{\theta}_X(\mathcal{A})(\varphi) = \bigoplus\big\{t \odot s \odot \varphi(a) \mid \text{there exist } (A, t) \in \mathcal{A} \text{ and } (a, s) \in A\big\},$$

where $\varphi \in C(X)$. On the other hand,

$$\begin{aligned} \theta_X I(h_X)h_{\overline{I}(X)}(\mathcal{A})(\varphi) &= I(h_X)h_{\overline{I}(X)}(\mathcal{A})(\overline{\varphi}) = h_{\overline{I}(X)}(\mathcal{A})(\overline{\varphi}h_X) \\ &= \bigoplus\big\{t \odot \overline{\varphi}h_X(A) \mid (A, t) \in \mathcal{A}\big\} \\ &= \bigoplus\big\{t \odot h_X(A)(\varphi) \mid (A, t) \in \mathcal{A}\big\} \\ &= \bigoplus\Big\{t \odot \bigoplus\{s \odot \varphi(a) \mid (a, s) \in A\} \mid (A, t) \in \mathcal{A}\Big\} \\ &= h_X\overline{\theta}_X(\mathcal{A})(\varphi). \end{aligned}$$

Our next result shows that, for the defined above metrization of the sets of idempotent measures, the natural transformations from this monads comprise of non-expanding maps. To this end, having in mind diagram (1), it is enough to prove that the map $\overline{\theta}_X$ is non-expanding.

Proposition 3.11 *For any X,*

$$\overline{\theta}_X = \exp(1_X \times m)\, u_{X\times[0,1]\times[0,1]} \exp \lambda_{X\times[0,1]}.$$

Proof Let $(a, t) \in \overline{\theta}(\mathcal{A})$, where $\mathcal{A} \in \overline{I}^2(X)$. Then there are $(A, t) \in \mathcal{A}$ and $(a, s) \in A$ such that $r = t \odot s$. Since

$$\lambda_{X\times[0,1]}(A, t) = \{((a, s'), t) \mid (a, s') \in A\},$$

we conclude that $((a, s), t) \in u_{X\times[0,1]\times[0,1]} \exp \lambda_{X\times[0,1]}$ and therefore

$$(a, r) = (a, s \odot t) \in \exp(1_X \times m)\, u_{X\times[0,1]\times[0,1]} \exp \lambda_{X\times[0,1]}.$$

Thus,

$$\overline{\theta}_X \subset \exp(1_X \times m)\, u_{X\times[0,1]\times[0,1]} \exp \lambda_{X\times[0,1]}.$$

The reverse inclusion is easy to check as well. □

Corollary 3.12 *The map $\overline{\theta}_X$ is non-expanding.*

Proof This is a consequence of Proposition 3.11 and the following facts:

- The maps $\lambda_{X\times[0,1]}$ and m_X are non-expanding (see Lemmas 3.9 and 3.7).
- The map $u_{X\times[0,1]\times[0,1]}$ is non-expanding.
- The functor exp preserves the class of non-expanding maps (see [16]). □

Finally, we obtain the following result.

Theorem 3.13 *The triple (I, η, θ) is a monad in the category $\mathfrak{FMS}(\cdot)$.*

4 Remarks and open questions

For the t-norm min, one can use the following fuzzy metric M on the segment $[-\infty, 0]$:

$$M(x, y, t) = e^{-|e^x - e^y|/t}$$

(see [6]).

If $*$ is the Łukasiewicz t-norm, then one can use the following fuzzy metric on the segment $[-\infty, 0]$:

$$M(x, y, t) = 1 - \left((2 + e^{\min\{x,y\}})^{-1} - (2 + e^{\max\{x,y\}})^{-1}\right)$$

(this is a modification of one example from [15]). The question for an arbitrary t-norm $*$ remains open.

Let $\overline{\mathbb{R}} = \mathbb{R}_{\max} \cup \{\infty\} = \mathbb{R} \cup \{-\infty, \infty\}$. In the sequel, $\otimes$ is used for min. A functional $\mu\colon C(X) \to \mathbb{R}$ is called a *max-min measure* if the following conditions are satisfied:

- $\mu(c_X) = c$;
- $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$;
- $\mu(c \otimes \varphi) = c \otimes \mu(\varphi)$

(see, e.g. [3] for details). By $J(X)$ we denote the set of all max-min measures on a compact Hausdorff space X. The set $J(X)$ is endowed with the weak*-topology.

Using arguments similar to the above, one can construct a fuzzy metric of the spaces $J(X)$ for a fuzzy metric space X. An ultrametrization of the spaces $J(X)$ for ultrametric X is given in [5]. It looks plausible that the methods of [8] can be applied also to the spaces of max-min measures to obtain their fuzzy ultrametrization.

Remark 4.1 One can also construct a fuzzy metric on the set of idempotent measures with compact support on noncompact fuzzy metric spaces.

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