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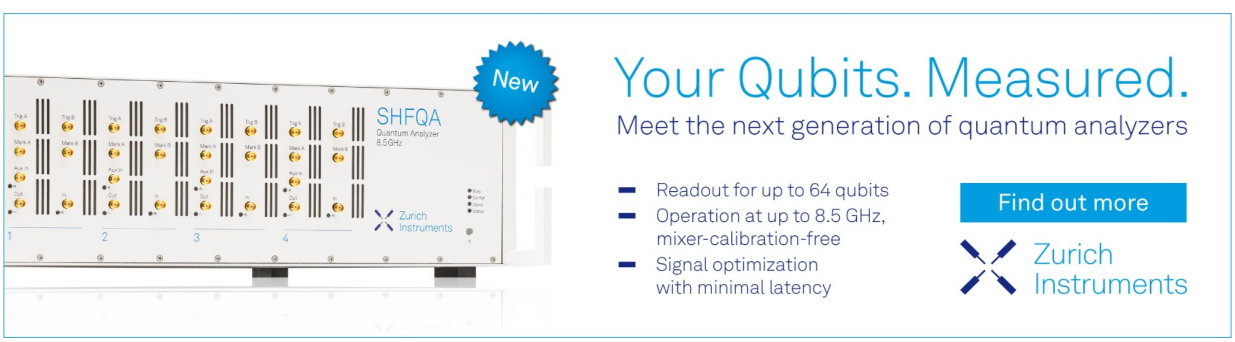

On the Typology of Quasi-Einstein Spaces

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Abstract. The paper treats a particular type of pseudo-Riemannian spaces, namely quasi-Einstein spaces with gradient defining vector. These spaces are a generalization of well-known Einstein spaces. There are three types of these spaces that permit locally geodesic mappings. We studied some geometric properties of every type.

INTRODUCTION

Let us study a pseudo-Riemannian space $V_n (n > 2)$, with a metric tensor g_{ij}. Here we construct an Einstein tensor in this space. The tensor is defined by a known expression:

$$E_{ij} \overset{def}{=} R_{ij} - \frac{R}{n} g_{ij},$$

where R_{ij} – Ricci tensor $R_{ij} \overset{def}{=} R^{\alpha}_{ij\alpha}$, R is a scalar curvature $R_{\alpha\beta} g^{\alpha\beta} = R$, R^h_{ijk} – Riemannian tensor. A defect of Einstein tensor [13] is a tensor D_{ij}, defined by an equation

$$E_{ij} - D_{ij} = 0.$$

When selecting a special type of tensor D_{ij}, one can select a particular type of special pseudo-Riemannian spaces. For example, if D_{ij} is a linear combination of metric tensor and covariant derivative of a certain vector, then taking into account coefficients of this combination, one can obtain $\varphi(Ric)$ spaces or Ricci solitons [4, 7]. When D_{ij} is represented by a simple bivector, called defining, then the space is quasi-Einstein [6]. Mapping is a main way for modeling of the above-mentioned spaces. We conducted a research aimed at conformal and geodesic mappings of pseudo-Riemannian spaces with various types of deformation tensor of Einstein tensor [2, 9, 10]. This work treats geodesic mappings of quasi-Einstein spaces with gradient defining vector. These spaces are subdivided into three types: main, particular and special. The obtained results were applied for a research on some geometric properties of spaces of every type.

BASIC EQUATIONS OF GEODESIC MAPPINGS THEORY

Bijection of points of pseudo-Riemannian spaces V_n with a metric tensor g_{ij} and $\bar{V}_n$ with a metric tensor $\bar{g}_{ij}$ is a geodesic mapping when every geodesic line V_n is transformed into a geodesic line $\bar{V}_n$. Pseudo-Riemannian spaces V_n and $\bar{V}_n$ that permit a geodesic mapping between them are called spaces in geodesic correspondence or belonging to a single geodesic class. In order to define pseudo-Riemannian spaces V_n and $\bar{V}_n$ as permitting bijective geodesic mappings there is a necessary and sufficient condition [17]

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \varphi_i \delta^h_j + \varphi_j \delta^h_i, \tag{1}$$

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or otherwise, taking into account a covariant constancy of a metric tensor,

$$\bar{g}_{ij},_k = 2\varphi_k\bar{g}_{ij} + \varphi_i\bar{g}_{jk} + \varphi_j\bar{g}_{ik}, \tag{2}$$

here φ_i is a certain (necessarily gradient) vector; $\Gamma^h_{ij}, \bar{\Gamma}^h_{ij}$ – Christoffel symbols V_n and $\bar{V}_n$ respectively; δ^h_i – Kronecker symbols; comma "," is a sign of covariant derivatives in respect to connectivity V_n [1].

Eqs. (1) and (2) are equivalent, they are necessary and sufficient conditions for bijective geodesic correspondence of pseudo-Riemannian spaces V_n and $\bar{V}_n$. The equations represent a necessary condition for a geodesic mapping:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \varphi_{ij}\delta^h_k - \varphi_{ik}\delta^h_j, \tag{3}$$

$$\bar{R}_{ij} = R_{ij} + (n-1)\varphi_{ij}, \tag{4}$$

here $\varphi_{ij} = \varphi_{i,\,j} - \varphi_i\varphi_j$; R^h_{ijk}, R_{ij} – Riemannian and Ricci tensors. Geodesic mapping that differs from homothety is called non-trivial. A certain pseudo-Riemannian space V_n permits non-trivial geodesic mapping when it contains a solution of system of differential equations in respect to tensor $a_{ij} = a_{ji} \neq cg_{ij}$ and vector $\lambda_i = \lambda_{,i} \neq 0$. It is a necessary and sufficient condition. This system is called a linear form of main equations. Linear form of main equations for geodesic mappings theory can be written as follows [17]

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{5}$$

$$n\lambda_{i,j} = \mu g_{ij} + a_{\alpha i}R^\alpha_j - a_{\alpha\beta}R^{\alpha\ \ \beta}_{\cdot\, ij\,\cdot}, \tag{6}$$

here $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$; $R^i_j = R_{\alpha j}g^{\alpha i}$; $R^{h\ \ k}_{\ ij} = R^h_{ij\alpha}g^{\alpha\ k}_{\ \cdot}$. It follows from the latter:

$$(n-1)\mu_{,i} = 2(n+1)\lambda_\alpha R^\alpha_i + a_{\alpha\beta}(2R^{\alpha\ \ \beta}_{\cdot\, i,\,\cdot} - R^{\alpha\beta}_{\ \ ,i}). \tag{7}$$

Solutions (2) and (5) are connected by relation

$$a_{ij} = e^{2\varphi}\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j};$$
$$\lambda_i = -e^{2\varphi}\bar{g}^{\alpha\beta}g_{\alpha i}g_\beta. \tag{8}$$

System of equations (5), (6) and (7) opens a possibility to answer a question: whether a certain pseudo-Riemannian space V_n permits a geodesic mapping onto a pseudo-Riemannian space $\bar{V}_n$. The problem is reduced to finding the integrability conditions of these equations and their differential extensions. This system is called a system of main equations of theory of geodesic mappings [11, 12]. Pseudo-Riemannian spaces $V_n(n > 2)$ are called quasi-Einstein spaces, when the following condition is true

$$R_{ij} = \frac{R}{n}g_{ij} + U_iU_j,$$

here U_i is defined as gradient vector, or otherwise

$$U_i = U_{,i} = \partial_i U.$$

The definition implies that vector U_i is a necessarily isotropic vector [6]. R is a scalar curvature, selected in a way that $R = R_{\alpha\beta}g^{\alpha\beta}$. The work [8], proves that when a quasi-Einstein space V_n permits non-trivial geodesic mappings, then for this space the following condition is true:

$$\lambda_{i,j} = \mu g_{ij} + \frac{R}{n(n-1)}a_{ij}, \tag{9}$$

or

$$\lambda_i - \nu U_i = 0. \tag{10}$$

According to the latter statement, quasi-Einstein spaces can be subdivided into three types:

1. Main type – when the equation (9) is true, while (10) is not true;
2. Particular type – when the equation (10) is true and the equation (9) is not;
3. Special type – when both equations (9) and (10) are true.

In the further discussion we are going to treat different types of quasi-Einstein spaces consequently.

QUASI-EINSTEIN SPACES OF THE MAIN TYPE

Let us treat a quasi-Einstein space of the main type, namely every quasi-Einstein space, which permits non-trivial geodesic mappings and where conditions (5), (9) are true. So far as it is proved in the work [8], then there are following conditions imposed on the invariant μ

$$\mu_{,i} = \frac{2R}{n(n-1)}, \tag{11}$$

and a scalar curvature R is a constant. Let us find a covariant derivative for the equation (8) and take into account the equation (2)

$$\lambda_{ij} = -e^{2\varphi}\varphi_{\alpha,j}\bar{g}^{\alpha\beta}g_{\beta i} + e^{2\varphi}\varphi_\alpha\varphi_\beta\bar{g}^{\alpha\beta}g_{ij} + e^{2\varphi}\varphi_j\varphi_\alpha\bar{g}^{\alpha\beta}g_{\beta i}, \tag{12}$$

here $\bar{g}^{ij}$ are elements of an inverse matrix for a metric tensor $\bar{V}_n$, of a space that corresponds in a geodesic sense to V_n. Let us substitute (10) in (12), and then, take into account (8) and multiply the result by $e^{-2\varphi}$

$$e^{-2\varphi}\mu g_{ij} + \frac{2R}{n(n-1)}\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j} = -\varphi_{\alpha,j}\bar{g}^{\alpha\beta}g_{\beta i} + \varphi_\alpha\varphi_\beta\bar{g}^{\alpha\beta}\bar{g}_{ij} + \varphi_j\varphi_\alpha\bar{g}^{\alpha\beta}g_{\beta i}. \tag{13}$$

Multiplying (13) by $g^{i\alpha}\bar{g}^{\beta k}$, we obtain

$$\varphi_{k,j} - \varphi_k\varphi_i = \bar{B}\bar{g}_{kj} - \frac{R}{n(n-1)}g_{kj}, \tag{14}$$

where $\bar{B} = \varphi_\alpha\varphi_\beta\bar{g}^{\alpha\beta} - e^{2\varphi}\mu$. $\bar{B}$ is some uniquely defined constant, so far as it is proved in the work [14]. Taking into account (14), we can re-write the equation (4) in the following way

$$\bar{R}_{ij} - \bar{B}\cdot(n-1)\bar{g}_{ij} = R_{ij} - \frac{R}{n}g_{kj}. \tag{15}$$

Eq. (15) and the definition of quasi-Einstein spaces imply the following statement

Theorem 1. *A geodesic mapping of pseudo-Riemannian quasi-Einstein spaces of the main type results in a generalized quasi-Einstein space and the following is true*

$$D_{ij} = \left(\bar{B}(n-1) - \frac{\bar{R}}{n}\right)\bar{g}_{ij}, \tag{16}$$

here $\bar{R}$ is a scalar curvature $\bar{V}_n$.

QUASI-EINSTEIN SPACES OF PARTICULAR TYPE

In the following discussion, we will turn our attention of quasi-Einstein spaces of the particular type. Let us treat quasi-Einstein spaces of the particular type. The following statement is true for them

$$\lambda_i = vU_i \tag{17}$$

and for convenience

$$U_i = s\lambda_i, \tag{18}$$

here $s = \frac{1}{v}$. Then, the above-mentioned expressions can be re-written in the following way

$$R_{ij} - \frac{R}{n}g_{ij} = S^2\lambda_i\lambda_j; \tag{19}$$

$$\lambda^\alpha a_{\alpha i} = \rho\lambda_i. \tag{20}$$

Let us differentiate (18)

$$U_{i,j} = S_j\lambda_i + S\lambda_{i,j}. \tag{21}$$

Alternating the latter

$$S_j\lambda_i - S_i\lambda_j = 0, \tag{22}$$

let us wrap (22) with a vector η^i selected in a such way that $\eta^\alpha\lambda_\alpha = 1$. Then, we arrive at

$$S_i = \gamma\lambda_i, \tag{23}$$

where $\gamma \overset{def}{=} S_\alpha\eta^\alpha$. Then, the equation (21) is re-written as follows

$$U_{i,j} = \gamma\lambda_i\lambda_j + S\lambda_{i,j}. \tag{24}$$

As far as the vector U_i is an isotropic vector, then the equation (17) implies that λ_i is an istropic vector too. Eqs. (19) and (24) can be transformed respectively to

$$\lambda_\alpha R^\alpha_i = \frac{R}{n}\lambda_i; \tag{25}$$

$$U_{\alpha,}{}^{\alpha} = S\lambda_{\alpha,}{}^{\alpha}, \tag{26}$$

here $\lambda_{\alpha,}{}^{\alpha} = \lambda_{\alpha\beta}g^{\alpha\beta}$.

Let us prove the following theorem:

Theorem 2. *The following conditions are true for the quasi-Einstein spaces of the particular type*

$$\lambda^\alpha_{,j}a_{\alpha i} = \overset{1}{\rho}\lambda_i\lambda_j + \rho\lambda_{i,j}. \tag{27}$$

Proof.

Let us differentiate (20) taking into account (5) and the fact that λ^i is an isotropic vector, then we arrive

$$\lambda_i\lambda_j + \lambda^\alpha_{,j}a_{\alpha i} = \rho_j\lambda_i + \rho\lambda_{i,j}. \tag{28}$$

Let us alternate, taking into account Lemma 1 from the work [14]:

$$\rho_j\lambda_i - \rho_i\lambda_j = 0. \tag{29}$$

Let us multiply by the vector η^i selected in such a way that $\eta^i\lambda_i = 1$. Then we get

$$\lambda^\alpha_{,j}a_{\alpha i} = (k-1)\lambda_i\lambda_j + \rho\lambda_{i,j}. \tag{30}$$

Here $k = \eta^\alpha\rho_\alpha$. Thus, the theorem is proved and we should note that $\overset{1}{\rho} = k - 1$.

It is well known [14] that tensor a_{ij} complies to the conditions

$$a_{\alpha\beta}T^{\alpha\beta}_{ij} = 0, \tag{31}$$

where

$$T^{\alpha\beta}_{ij} = \delta^\alpha_i R^\beta_j - \delta^\alpha_j R^\beta_i. \tag{32}$$

Pseudo-Riemannian space which contain

$$a_{\alpha\beta}T^{\alpha\beta}_{ij,k} = 0, \tag{33}$$

will be called geodesic Ricci-symmetrical. Let us treat tensor

$$T^{\alpha\beta}_{ijkl} = \delta^\alpha_j R^\beta_{ikl} + \delta^\alpha_k R^\beta_{ilj} + \delta^\alpha_l R^\beta_{ijk}. \tag{34}$$

It is true for tensor a_{ij}

$$a_{\alpha\beta}T^{\alpha\beta}_{ijkl} = 0. \tag{35}$$

Let us call spaces, which comply to the conditions

$$a_{\alpha\beta}T^{\alpha\beta}_{ijkl,m} = 0, \tag{36}$$

geodesic symmetrical spaces.

Wrapping (34) by indices i, j, we obtain a statement

Lemma 1. *Geodesic symmetrical spaces are geodesic Ricci-symmetrical spaces.*

The following statement is true.

Theorem 3. *A quasi-Einstein space V_n is a space of the particular type with a sufficiency and necessity when V_n is a geodesic Ricci-symmetrical space and the equation (9) should not be true for it.*

Proof.
Let us differentiate (31) taking into account (33) and (5). We obtain

$$\lambda_\alpha R^\alpha_i g_{jk} + \lambda_j R_{ik} - \lambda_\alpha R^\alpha_j g_{ik} - \lambda_i R_{jk} = 0. \tag{37}$$

By substitution of (25) in (37), we get

$$\lambda_i \left(R_{jk} - \frac{R}{n} g_{kj} \right) - \lambda_j \left(R_{ik} - \frac{R}{n} g_{ik} \right) = 0. \tag{38}$$

The latter equation proves the theorem is true. Whether the obtain result is a sufficient condition, we can determine by direct substitution of formula e defining the particular type of quasi-Einstein spaces into (33).

The research on this issue was carried out by application of the methods developed in the works [18, 19].

Theorem 4. *There is no geodesic symmetrical pseudo-Rieamnnian spaces of a constant scalar curvature, belonging to the particular type.*

Proof.
Eq. (35), taking into account (5), (36), implies

$$\lambda_\alpha R^\alpha_{jkl} g_{im} + \lambda_\alpha R^\alpha_{jli} g_{km} + \lambda_i R_{mjkl} + \lambda_k R_{mjli} + \lambda_\alpha R^\alpha_{jik} g_{lm} + \lambda_l R_{mjik} = 0. \tag{39}$$

Wrapping (39) by indices i, m and, taking into account the definition of a quasi-Einstein space, we can formulate the following equation

$$\lambda_\alpha R^\alpha_{ijk} = \frac{R}{n(n-1)} (\lambda_k g_{ij} - \lambda_j g_{ik}). \tag{40}$$

Integrability conditions of equations (5) take the following shape

$$a_{\alpha i} R^\alpha_{jkl} + a_{\alpha j} R^\alpha_{ikl} = \lambda_{li} g_{jk} + \lambda_{lj} g_{ik} - \lambda_{ki} g_{jl} - \lambda_{kj} g_{il}. \tag{41}$$

Let us multiply (41) by λ^l and wrap it by index l. Then taking into account that vector λ_i is an isotropic vector, we obtain

$$a^\alpha_i R^\beta_{kj\alpha} \lambda_\beta + a^\alpha_j R^\beta_{ki\alpha} \lambda_\beta = -\lambda_{ki} \lambda_j - \lambda_{kj} \lambda_i. \tag{42}$$

Substituting (40) and regrouping the members, we get

$$\lambda_i \left(\frac{R\rho}{n(n-1)} g_{jk} - \frac{R}{n(n-1)} a_{jk} + \lambda_{jk} \right) + \lambda_j \left(\frac{R\rho}{n(n-1)} g_{ik} - \frac{R}{n(n-1)} a_{ik} + \lambda_{ik} \right) = 0. \tag{43}$$

Alternating (43) by indices i, k. Reassigning the indices j and k and adding the result to (43), we arrive at

$$\lambda_{ij} = -\frac{R\rho}{n(n-1)} g_{ij} + \frac{R}{n(n-1)} a_{ij}. \tag{44}$$

So the theorem is proved.

CONCLUSIONS

Pseudo-Riemannian quasi-Einstein spaces hold an important position in the theory of geodesic mappings of generalized spaces. They are a direct generalization for Einstein spaces and as such find numerous applications in mechanics and physics. Every quasi-Einstein space is classified into three types according to their properties in relation to geodesic mappings. It is proved that equations, which define non-trivial geodesic mappings, are analogous to corresponding equations of Einstein spaces. There are some types which permit a notable deviation from these analogies [3, 5, 15, 20]. The directions for future research are study on pseudo-Riemannian spaces of small dimensions, construction of classes of geodesically correspondent spaces, and of geodesic mapping "in general."

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