

Geodesic mappings of compact quasi-Einstein spaces, II

V. Kiosak, A. Savchenko, O. Latysh

Abstract. The paper treats geodesic mappings of quasi-Einstein spaces with gradient defining vector.

Previously the authors defined three types of these spaces. In the present paper it is proved that there are no quasi-Einstein spaces of special type. It is demonstrated that quasi-Einstein spaces of main type are closed with respect to geodesic mappings. The spaces of particular type are proved to be geodesic D -symmetric spaces.

Анотація. Робота продовжує дослідження майже ейнштейнових псевдоріманових просторів з градієнтним задаючим вектором. При цьому дефект тензора Ейнштейна вважається відмінним від нуля, тобто досліджуються простори відмінні від просторів Ейнштейна.

В статті вивчаються нетривіальні геодезичні відображення майже ейнштейнових просторів за допомогою лінійної форми основних рівнянь теорії геодезичних відображень.

Відомо три типи майже ейнштейнових просторів з градієнтним задаючим вектором, що допускають геодезичні відображення: основний тип, спеціальний тип та особливий. Для просторів основного типу доведена їхня замкненість відносно нетривіальних геодезичних відображень, тобто доведено, що майже ейнштейнові простори основного типу дозволяють нетривіальні геодезичні відображення лише на майже ейнштейнові простори основного типу. Також показано, що просторів спеціального типу не існує, а для просторів особливого типу показано, що їх скалярна кривина не може бути сталою.

Степінь мобільності майже ейнштейнових просторів особливого типу не перевищує двох. Відомі типи псевдоріманових просторів, що мають степінь мобільності два – субпроективні простори Кагана та простори L_n – не можуть бути майже ейнштейновими просторами особливого типу. Не відносяться до особливого типу і простори, в яких лінійна система

UDC 514.765.1+512.813.4

Keywords: pseudo-Riemannian space, quasi-Einstein space, geodesic mapping

Ключові слова: псевдо-рімановий простір, квазі-ейнштейновий простір, геодезичне відображення

DOI: <http://dx.doi.org/10.15673/tmgc.v14i1.1936>

основних рівнянь має канонічний розв'язок. Для просторів особливого типу показано, коли вони належать до геодезично D -симетричних просторів.

Таким чином, існує два типи майже ейнштейнових просторів з градієнтним задаючим вектором, що допускають нетривіальні геодезичні відображення. Для основного типу скалярна кривина стала, а для особливого – ні. Приклади вказаних просторів слід шукати серед просторів $V(K)$ А. С. Солодовнікова та просторів Вренчану.

Дослідження ведуться локально в тензорній формі, без обмежень на сигнатуру та знак метричного тензора.

«Теореми зникнення» для компактних просторів можуть бути доведені з використанням результатів попередньої роботи.

1. INTRODUCTION

We will consider a pseudo-Riemannian space $V_n (n > 2)$ with a metric tensor g_{ij} . In this space we construct an Einstein tensor, which is defined by the following expression:

$$E_{ij} \stackrel{def}{=} R_{ij} - \frac{R}{n} g_{ij}, \quad (1.1)$$

where R_{ij} is the corresponding Ricci tensor, $R_{ij} \stackrel{def}{=} R_{ij\alpha}^{\alpha}$, R the scalar curvature $R_{\alpha\beta} g^{\alpha\beta} = R$, R_{ijk}^h the Riemannian tensor.

The tensor D_{ij} defined by

$$E_{ij} - D_{ij} = 0, \quad (1.2)$$

is called the *defect* of Einstein tensor [3].

In what follows we will treat pseudo-Riemannian spaces V_n , where

$$D_{ij} \neq 0,$$

or in other words the spaces distinct from Einstein spaces.

An Einstein tensor is an inner object of pseudo-Riemannian space V_n , and thus it is defined by a metric tensor.

By defining special type of tensor D_{ij} we can differentiate several types of special pseudo-Riemannian spaces. For example, when D_{ij} is a linear combination of a metric tensor and a covariant derivative of a vector, depending on the coefficients of this combination, we get either $\varphi(\text{Ric})$ or Ricci solitons [2].

By imposing some additional conditions on the space, we are able to obtain various special spaces.

When D_{ij} is a simple bivector, which is called “defining”, then we face a quasi-Einstein space [6].

A bijection between points of pseudo-Riemannian spaces V_n with a metric tensor g_{ij} and \bar{V}_n with a metric tensor \bar{g}_{ij} is called a *geodesic mapping*, when every geodesic line of V_n transforms into a geodesic line of \bar{V}_n .

The works [4, 5, 7] are devoted to the research on geodesic mappings of quasi-Einstein spaces with gradient defining vector. The above-mentioned spaces are divided into three types: main, particular and special [1].

In the present paper we obtain local results which can be applied to the study on compact quasi-Einstein spaces of the main type “in general” [11], and the paper [17] solved the same issues in respect to compact quasi-Einstein spaces of a constant scalar curvature [18].

There are numerous works referring to the topics of geodesic and conformal mappings and their application in general relativity theory. It underlines the importance of the topic of this paper [3, 8].

However, the plentiful works omitted the spaces of special type due to the absence of examples of these spaces.

This paper aims at the study of geodesic mappings of quasi-Einstein spaces of a special type and at the clarification of the tensor characteristics of spaces of other types.

2. GEODESIC MAPPINGS OF THE QUASI-EINSTEIN SPACES

Suppose that there exists a solution of the following equation in V_n :

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \quad (2.1)$$

with respect to a symmetrical tensor $a_{ij} \neq c g_{ij}$ and a vector λ_i , $\lambda_i = \lambda_{,i} \neq 0$, [10, 15].

Here the comma «,» is a sign of a covariant derivative in respect to the connection of V_n .

Equation (2.1) is called a *linear form* of the main equations of the theory of geodesic mappings.

An existence of non-trivial solutions of equations (2.1) with respect to a tensor a_{ij} and a vector λ_i is a necessary and sufficient condition for the space V_n to permit non-trivial geodesic mappings [14, p. 108].

Analyzing the integrability conditions of equations (2.1) we can see that the tensor a_{ij} as well as the Ricci tensor both comply to the following condition:

$$a_{i\alpha} R_j^\alpha - a_{\alpha j} R_i^\alpha = 0.$$

Meanwhile, taking into account (1.1) and (1.2) we get that

$$a_{i\alpha} D_j^\alpha - a_{\alpha j} D_i^\alpha = 0, \quad (2.2)$$

where $R_i^h = R_{\alpha i} g^{\alpha h}$, $D_i^h = D_{\alpha i} g^{\alpha h}$, and g^{ij} are elements of the matrix inverse to g_{ij} .

Differentiate the latter equation and taking into account (2.1) we obtain

$$\lambda_i D_{jk} + \lambda_\alpha D_j^\alpha g_{ik} - \lambda_j D_{ik} - \lambda_\alpha D_i^\alpha g_{jk} + a_{\alpha i} D_{j,k}^\alpha - a_{\alpha j} D_{i,k}^\alpha = 0. \quad (2.3)$$

The equation (2.2) can be rewritten as follows

$$a_{\alpha\beta} T_{ij}^{\alpha\beta} = 0. \quad (2.4)$$

Here

$$T_{ij}^{\alpha\beta} = \delta_i^\alpha D_j^\beta - \delta_j^\alpha D_i^\beta. \quad (2.5)$$

Pseudo-Riemannian spaces, where the following conditions are true

$$a_{\alpha\beta} T_{ij,k}^{\alpha\beta} = 0, \quad (2.6)$$

are called *geodesic D-symmetric*.

In geodesic *D-symmetric* pseudo-Riemannian spaces equations (2.3) can also be written down as follows:

$$\lambda_i D_{jk} + \lambda_\alpha D_j^\alpha g_{ik} - \lambda_j D_{ik} - \lambda_\alpha D_i^\alpha g_{jk} = 0. \quad (2.7)$$

Take into account that the tensor D_{ij} satisfies the following condition: $D_{\alpha\beta} g^{\alpha\beta} = 0$. Then, we can wrap (2.7) and arrive at

$$\lambda_\alpha D_i^\alpha = 0. \quad (2.8)$$

Then (2.7) can be written in the following form:

$$\lambda_i D_{jk} - \lambda_j D_{ik} = 0. \quad (2.9)$$

Multiplying (2.9) by $\lambda^i = \lambda_\alpha g^{\alpha i}$, wrapping it by index i , and taking into account (2.8), we get

$$\lambda_\alpha \lambda^\alpha D_{jk} = 0.$$

Multiplying further (2.9) by D_m^j we obtain:

$$\lambda_i D_{\alpha k} D_m^\alpha = 0.$$

Thus, if a geodesic *D-symmetric* space is distinct from Einstein space (i.e. $D \neq 0$) and permits a non-trivial geodesic mappings, then the vector λ_i is isotropic, that is

$$\lambda_\alpha \lambda^\alpha = 0,$$

and

$$R_{\alpha i} R_j^\alpha - \frac{2R}{n} R_{ij} + \left(\frac{R}{n}\right)^2 g_{ij} = 0. \quad (2.10)$$

Equation (2.10) is a characteristic of these spaces of «internal» character. Note that the equation (2.10) is a generalization of the algebraic part of known conditions of Rainich [12, p.55].

As far as $\lambda_i \neq 0$, we can select a vector ξ^i in such a way, that $\xi^\alpha \lambda_\alpha = 1$.

Then, from (2.9) we can obtain the following

$$D_{ij} = \tau_i \lambda_j, \quad (2.11)$$

where $\tau_i = \xi^\alpha D_{\alpha i}$. Multiplying the latter by ξ^i , we arrive at

$$\tau_i = w\lambda_i$$

and (2.11) can be rewritten as follows:

$$D_{ij} = w\lambda_i\lambda_j, \tag{2.12}$$

where $w = \xi^\alpha \xi^\beta D_{\alpha\beta}$.

Substituting (2.12) into (2.3), we see that conditions (2.6) are satisfied. Thus we have proved the following statement.

Theorem 2.1. *In order to define a pseudo-Riemannian space as geodesic D -symmetric, it is necessary and sufficient that the conditions (2.12) hold in that space.*

On the other hand, pseudo-Riemannian spaces $V_n (n > 2)$ satisfying the following conditions:

$$D_{ij} = u_i u_j, \tag{2.13}$$

where u_i is a gradient vector by definition, are called *quasi-Einstein spaces*.

It is proved in [11] that if a quasi-Einstein space V_n permits non-trivial geodesic mappings, then the following conditions hold for a vector λ_i

$$\lambda_{i,j} = \mu g_{ij} + \frac{R}{n(n-1)} a_{ij}, \tag{2.14}$$

or

$$u_i = v\lambda_i, \tag{2.15}$$

where

$$\mu_{,i} = \frac{2R}{n(n-1)} \lambda_i, \tag{2.16}$$

and v is some invariant.

Thus, quasi-Einstein spaces, that permit non-trivial geodesic mappings, are divided into three types [5]:

- (1) *main type* if condition (2.14) hold, while (2.15) fails;
- (2) *particular type* if condition (2.15) holds, while (2.14) fails;
- (3) *special type* if both (2.14) and (2.15) hold.

Not that particular and special spaces are geodesic D -symmetric spaces. Moreover, in order to define a geodesic D -symmetric space as a quasi-Einstein space, it is necessary, that a vector $\sqrt{w}\lambda_i$ is gradient.

3. QUASI-EINSTEIN SPACES OF SPECIAL TYPE

We will consider here geodesic mappings of quasi-Einstein spaces for which both equations (2.14), (2.15) are true.

The following statement holds:

Theorem 3.1. *There are no quasi-Einstein spaces of special type.*

Proof. The vector λ_i of quasi-Einstein spaces of the special type is isotropic, namely:

$$\lambda_\alpha \lambda^\alpha = 0. \quad (3.1)$$

Differentiating this equation and taking into account (2.14) we obtain

$$\mu \lambda_i + \frac{R}{n(n-1)} a_{i\alpha} \lambda^\alpha = 0. \quad (3.2)$$

Then, as far as $R = \text{const}$

$$\begin{aligned} \mu_j \lambda_i + \mu \lambda_{i,j} + \frac{R}{n(n-1)} \lambda_i \lambda_j + \\ + \frac{R}{n(n-1)} \cdot \mu a_{ij} + \left(\frac{R}{n(n-1)} \right)^2 a_{\alpha i} a_j^\alpha = 0. \end{aligned}$$

Substituting (2.14) and (2.16) we obtain:

$$\frac{3R}{n(n-1)} \lambda_i \lambda_j + \mu^2 g_{ij} + \frac{2R}{n(n-1)} \cdot \mu a_{ij} + \left(\frac{R}{n(n-1)} \right)^2 a_{\alpha i} a_j^\alpha = 0.$$

Differentiating this identity and using (3.2) we get:

$$R(\lambda_i A_{jk} + \lambda_j A_{ik} + \lambda_k A_{ij}) = 0,$$

where

$$A_{ij} = \mu g_{ij} + \frac{R}{n(n-1)} a_{ij}.$$

In other words, the equations (2.14) imply that

$$R(\lambda_i \lambda_{j,k} + \lambda_j \lambda_{i,k} + \lambda_k \lambda_{i,j}) = 0. \quad (3.3)$$

Suppose that $R \neq 0$. Then, it follows from (3.3) and (2.14) that

$$n\mu + \frac{R}{n(n-1)} a = 0, \quad (3.4)$$

where $a = a_{\alpha\beta} g^{\alpha\beta}$.

From (2.1) we see that $a_{,i} = 2\lambda_i$. Then (3.4) and (3.1) also imply the following identity:

$$2 \frac{R}{n(n-1)} \cdot n + 2 \frac{R}{n(n-1)} = 0. \quad (3.5)$$

The latter contradicts to the statement that

$$R \neq 0.$$

Hence the above-mentioned spaces have zero scalar curvature. Together with (3.2) the latter imply that

$$\mu = 0,$$

whence (2.14) can be rewritten as follows

$$\lambda_{i,j} = 0.$$

Substituting (2.15) into (2.13) and taking into account that $R = 0$ we obtain

$$R_{ij} = v^2 \lambda_i \lambda_j,$$

whence

$$R_{ij,k} = 2vv_{,k} \lambda_i \lambda_j.$$

In other words

$$R_{ij,k} = \rho_{,k} R_{ij}, \tag{3.6}$$

where $\rho \stackrel{def}{=} \ln^2 v$

The spaces satisfying conditions (3.6) are called *recurrent*. They do not admit non-trivial geodesic mappings as long as they are not spaces of a constant curvature [14, p.131]. This proves Theorem 3.1. \square

4. QUASI-EINSTEIN SPACES OF MAIN TYPE

Consider no quasi-Einstein spaces of main type. These are quasi-Einstein spaces that permit non-trivial geodesic mappings and satisfying conditions (2.14) and (2.16) hold.

Pseudo-Riemannian spaces V_n and \bar{V}_n permitting geodesic mappings one onto other are called *spaces in a geodesic correspondence* or the ones *belonging to the same geodesic class*.

Due to [14, p.107] pseudo-Riemannian spaces V_n and \bar{V}_n permit geodesic mappings one onto other if and only if

$$\bar{\Gamma}_{ij}^h = \Gamma_{ij}^h + \varphi_i \delta_j^h + \varphi_j \delta_i^h, \tag{4.1}$$

or, equivalently, taking into account that metric tensor is covariantly constant when

$$\bar{g}_{ij,k} = 2\varphi_k \bar{g}_{ij} + \varphi_i \bar{g}_{jk} + \varphi_j \bar{g}_{ik}, \tag{4.2}$$

where φ_i is a certain vector (which is necessarily gradient), Γ_{ij}^h , $\bar{\Gamma}_{ij}^h$ are Christoffel symbols of V_n and \bar{V}_n relatively, and δ_i^h is the Kronecker symbol.

Solutions of (4.2) and (2.1) are the following relation

$$a_{ij} = e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j} \tag{4.3}$$

$$\lambda_i = -e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta}, \quad (4.4)$$

where \bar{g}^{ij} are elements of a matrix inverse to the metric tensor \bar{V}_n of a space which is in a geodesic correspondence with V_n .

Applying covariant differentiation of equations (4.4) and taking into account (4.2) we get

$$\lambda_{ij} = -e^{2\varphi} \varphi_{\alpha,j} \bar{g}^{\alpha\beta} g_{\beta i} + e^{2\varphi} \varphi_{\alpha} \varphi_{\beta} \bar{g}^{\alpha\beta} g_{ij} + e^{2\varphi} \varphi_j \varphi_{\alpha} \bar{g}^{\alpha\beta} g_{\beta i}. \quad (4.5)$$

Substituting further (2.14) into (4.5), taking into account (4.3), and multiplying by $e^{-2\varphi}$, we obtain:

$$\begin{aligned} e^{-2\varphi} \mu g_{ij} + \frac{2R}{n(n-1)} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j} = \\ = -\varphi_{\alpha,j} \bar{g}^{\alpha\beta} g_{\beta i} + \varphi_{\alpha} \varphi_{\beta} \bar{g}^{\alpha\beta} \bar{g}_{ij} + \varphi_j \varphi_{\alpha} \bar{g}^{\alpha\beta} g_{\beta i}. \end{aligned} \quad (4.6)$$

Multiplying (4.6) by $g^{i\alpha} \bar{g}^{\beta k}$ we get

$$\varphi_{k,j} - \varphi_k \varphi_j = \bar{B} \bar{g}_{kj} - \frac{R}{n(n-1)} g_{kj}, \quad (4.7)$$

where $\bar{B} = \varphi_{\alpha} \varphi_{\beta} \bar{g}^{\alpha\beta} - e^{2\varphi} \mu$. It is proved in [11] that in this case \bar{B} is some uniquely defined constant.

The following equations are necessary conditions for a geodesic mapping:

$$\begin{aligned} \bar{R}_{ijk}^h &= R_{ijk}^h + \varphi_{ij} \delta_k^h - \varphi_{ik} \delta_j^h, \\ \bar{R}_{ij} &= R_{ij} + (n-1) \varphi_{ij}, \end{aligned} \quad (4.8)$$

where $\varphi_{ij} = \varphi_{i,j} - \varphi_i \varphi_j$, R_{ijk}^h and R_{ij} are Riemannian and Ricci tensors respectively.

Substituting (4.7) into (4.8) and taking into account (2.13), we obtain

$$\bar{R}_{ij} - \bar{B}(n-1) \bar{g}_{ij} = u_i u_j,$$

or, in other form

$$\bar{E}_{ij} = \left(\bar{B}(n-1) - \frac{\bar{R}}{n} \right) \bar{g}_{ij} + u_i u_j.$$

When $\bar{B} = \frac{\bar{R}}{n(n-1)}$, then \bar{V}_n is a quasi-Einstein space of the same type and with the same defining vector.

The following statement is true.

Theorem 4.1. *Suppose that for a geodesic mapping between quasi-Einstein spaces of the main type the following identity holds: $\bar{B} = \frac{\bar{R}}{n(n-1)}$. Then the Einstein tensor is preserved under such map.*

Let us note, that mappings preserving Einstein tensor were studied in [9, 13].

5. QUASI-EINSTEIN SPACES OF PARTICULAR TYPE

The *mobility* of a space with respect to geodesic mappings is a number of spaces onto which the given space permits non-trivial geodesic mappings.

The *degree of mobility* r is a number of non-trivial solutions of equations (2.1) and their differential extensions.

Suppose the following condition holds for a vector λ_i from the equations (2.1):

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij}. \tag{5.1}$$

then these spaces are denoted by $V_n(B)$, where B and μ are some invariants [14, p. 108].

For pseudo-Riemannian spaces having $r > 2$ the conditions (5.1) are necessarily hold with $B = \text{const}$ and

$$\mu_{,i} = 2B\lambda_i.$$

Here we proved that quasi-Einstein spaces cannot belong to spaces $V_n(B)$ with $B = \text{const}$. Thus, their degree of mobility in respect to geodesic mappings cannot exceed 2.

The known spaces having a degree of mobility equal to 2 with respect to geodesic mappings are conformal flat spaces and spaces L_n . Both above-mentioned types of spaces permit canonical solutions of equations (2.1), see [16].

The solutions for a system (2.1) are called *canonical* whenever

$$a_{ij} = \frac{1}{\tau} g_{ij} + \frac{2}{\tau} R_{ij}. \tag{5.2}$$

Further, we consider the following problem: do quasi-Einstein spaces of particular type permit canonical solutions? Namely we will prove the following statement.

Theorem 5.1. *There is no quasi-Einstein space V_n which permits canonical solutions.*

Proof. For quasi-Einstein spaces of the particular type the equations (5.2) can be rewritten as follows:

$$a_{ij} = \left(\frac{1}{\tau} + \frac{2R}{n} \right) g_{ij} + \frac{2}{\tau} \cdot v^2 \lambda_i \lambda_j, \tag{5.3}$$

where $\frac{3}{\tau} = \frac{1}{\tau} + \frac{2R}{n}$, $\frac{4}{\tau} = \frac{2}{\tau} \cdot v^2$.

Applying covariant differentiation of (5.3) we get:

$$\lambda_i g_{jk} + \lambda_j g_{ik} = \frac{3}{\tau_k} g_{ij} + \frac{4}{\tau_k} \lambda_i \lambda_j + \frac{4}{\tau} \lambda_{i,k} \lambda_j + \frac{4}{\tau} \lambda_i \lambda_{j,k}. \quad (5.4)$$

Wrapping by indices i, j and taking into account that the vector λ_i is isotropic we further obtain that

$$2\lambda_k = n \frac{3}{\tau_k}.$$

Hence we can rewrite (5.4) as follows

$$\lambda_i A_{jk} + \lambda_j A_{ik} = \frac{2}{n} \lambda_k g_{ij} + \frac{4}{\tau_k} \lambda_i \lambda_j, \quad (5.5)$$

where $A_{jk} = g_{jk} - \frac{4}{\tau} \lambda_{j,k}$. Then $\lambda^\alpha A_{\alpha j} = \lambda_j$ and $\frac{4}{\tau} \lambda^\alpha = 2$.

Multiplying (5.5) by vector λ_i and wrapping by index i we finish the proof of theorem. \square

Pseudo-Riemannian spaces satisfying the conditions

$$a_{\alpha i} R_{j,k}^\alpha - a_{\alpha j} R_{i,k}^\alpha = 0$$

are called *geodesic Ricci-symmetric*, while spaces with

$$a_{\alpha\beta} T_{ijkl,m}^{\alpha\beta} = 0$$

are called *geodesic symmetrical*.

Here

$$T_{ijkl}^{\alpha\beta} = \delta_j^\alpha R_{ikl}^\beta + \delta_k^\alpha R_{ilj}^\beta + \delta_l^\alpha R_{ijk}^\beta.$$

It is well-known, [5], that geodesic symmetrical spaces are geodesic Ricci-symmetric spaces. Moreover, a quasi-Einstein space is a space of particular type if and only if it is geodesic Ricci-symmetric also satisfying conditions (2.14). Then, taking into account Theorem 4.1, the condition of falsity of (2.14) can be abandoned. Finally, geodesic symmetrical spaces belong to particular type whenever their scalar curvature is not constant.

For quasi-Einstein spaces with constant curvature the classes of geodesic D -symmetrical and geodesic Ricci-symmetric spaces coincide.

6. CONCLUSION

In the present paper it is proved that there exist only two types of quasi-Einstein spaces permitting geodesic mappings.

Quasi-Einstein spaces of main type belong to the spaces $V_n(B)$. They are closed in respect to geodesic mappings, or in other words, they permit geodesic mappings only onto quasi-Einstein spaces of main type. Moreover, Einstein tensor is invariant under such mappings.

Quasi-Einstein spaces of special type do not exist.

Quasi-Einstein spaces of particular type have a degree of mobility with respect to geodesic mappings not exceeding 2. On the other hand, these spaces do not permit canonical solutions of the main system of equations.

REFERENCES

- [1] L. E. Evtushik, V. A. Kiosak, J. Mikeš. The mobility of Riemannian spaces with respect to conformal mappings onto Einstein spaces. *Russian Mathematics*, 54(8):29–33, 2010, doi: 10.3103/S1066369X10080049.
- [2] I. Hinterleitner, V. Kiosak. $\varphi(\text{Ric})$ -vector fields on conformally flat spaces. *Proceedings of American Institute of Physics*, 1191:98–103, 2009, doi: 10.1063/1.3275604.
- [3] V. Kiosak, A. Savchenko, O. Gudyreva. On the conformal mappings of special quasi-Einstein spaces. In *AIP Conference Proceedings*, volume 2164 (040001), 2019, doi: 10.1063/1.5130793.
- [4] V. Kiosak, A. Savchenko, A. Kamienieva. Geodesic mappings of compact quasi-Einstein spaces with constant scalar curvature. In *AIP Conference Proceedings*, volume 2302 (040002), 2020, doi: 10.1063/5.0033661.
- [5] V. Kiosak, A. Savchenko, S. Khniunin. On the typology of quasi-Einstein spaces. In *AIP Conference Proceedings*, volume 2302 (040003), 2020, doi: 10.1063/5.0033700.
- [6] V. A. Kiosak. On the conformal mappings of quasi-Einstein spaces. *Journal of Mathematical Sciences, (United States)*, 184(1):12–18, 2012, doi: 10.1007/s10958-012-0848-6.
- [7] V. A. Kiosak, G. V. Kovalova. Geodesic mappings of quasi-Einstein spaces with a constant scalar curvature. *Mat. Stud.*, 53(2):212–217, 2020, doi: 10.30970/ms.53.2.212-217.
- [8] V. A. Kiosak, V. S. Matveev. There are no conformal Einstein rescalings of complete pseudo-Riemannian Einstein metrics. *C. R. Math. Acad. Sci. Paris*, 347(17-18):1067–1069, 2009, doi: 10.1016/j.crma.2009.06.017.
- [9] V. A. Kiosak, V. S. Matveev. There exist no 4-dimensional geodesically equivalent metrics with the same stress-energy tensor. *J. Geom. Phys.*, 78:1–11, 2014, doi: 10.1016/j.geomphys.2014.01.002.
- [10] V. A. Kiosak, V. S. Matveev, J. Mikeš, I. G. Shandra. On the degree of geodesic mobility for Riemannian metrics. *Mathematical Notes*, 87(3-4):586–587, 2010, doi: 10.1134/S0001434610030375.
- [11] V. A. Kiosak, A. G. Savchenko, G. Kovalova. Geodesic mappings of compact quasi-Einstein spaces, I. *Proc. Int. Geom. Cent.*, 13(1):35–48, 2020, doi: 10.15673/t-mgc.v13i1.1711.
- [12] D. Kramer, H. Stephani, M. Maccollum, E. Herlt. *Exact solutions of the Einstein field equations*. Leutscher Verlag der Wissenschaften, Berlin, 1980, doi: 10.1017/CBO9780511535185.
- [13] O. Lesechko, O. Latysh, A. Kamienieva. Models of mechanical systems preserving the Weyl tensor. In *AIP Conference Proceedings*, volume 2164 (040002), 2019, doi: 10.1063/1.5130794.
- [14] J. Mikeš, V. Kiosak, A. Vanžurová. *Geodesic mappings of manifolds with affine connection*. Palacký University Olomouc, Olomouc, 2008.
- [15] N. S. Sinyukov. *Geodesic mappings of Riemannian spaces*. “Nauka”, Moscow, 1979 (in Russian).
- [16] H. N. Sinyukova. Geodesic uniqueness in the large of some generally recurrent Riemannian spaces. *J. Math. Sci.*, 177(5):710–715, 2011, doi: 10.1007/s10958-011-0500-x.

- [17] Y. Vashpanov, O. Olshevska, O. Lesechko. Geodesic mappings of spaces with $\varphi(\text{Ric})$ vector fields. In *AIP Conference Proceedings*, volume 2302 (040010), 2020, doi: 10.1063/5.0033965.
- [18] M. Zarichnyi, A. Savchenko, V. Kiosak. Strong topology on the set of persistence diagrams. In *AIP Conference Proceedings*, volume 2164 (040006), 2019, doi: 10.1063/1.5130798.

Received: January 12, 2021, accepted: March 19, 2021.

V. Kiosak

ODESA STATE ACADEMY OF CIVIL ENGINEERING AND ARCHITECTURE,
DIDRIHSON ST., 4, ODESA, 65029, UKRAINE

Email: kiosakv@ukr.net

ORCID: orcid.org/0000-0002-7433-6709

A. Savchenko

KHERSON STATE UNIVERSITY,
UNIVERSITY STREET 27, KHERSON, UKRAINE 73003, UKRAINE

Email: savchenko.o.g@ukr.net

ORCID: orcid.org/0000-0003-4687-5542

O. Latysh

ODESA NATIONAL UNIVERSITY MARITIME ACADEMY,
DIDRIHSON ST., 8, ODESA, 65029, UKRAINE

Email: latysh@ukr.net

ORCID: orcid.org/0000-0001-8914-2889