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Geometric interpretation and visualization of particular geometric concepts at metric spaces study

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Abstract. The paper considers the issues of studying method of geometric properties of metric spaces. These questions arise when students learn the basic concepts of the metric spaces theory. Difficulty in the concepts understanding arises due to the lack of the geometric interpretation or appropriate visualization. To build a geometric interpretation of rectilinear and flat placement of points of metric space, it is proposed to build the appropriate analogues in two-dimensional and three-dimensional arithmetic Euclidean spaces. To visualize these concepts, it is proposed to use a dynamic geometric environment GeoGebra 3D. This approach allows to demonstrate both the similarity of individual geometric concepts of metric space with the corresponding concepts of Euclidean geometry, and cases of the "non-Euclidean". The study is useful for teachers and students of higher education institutions majoring in physics and mathematics. Some examples can be used in the study of basic geometric concepts by students of secondary education, in-depth study of mathematics and in various types of informal education.

1. Introduction

For the first time, students of higher education institutions get acquainted with metric spaces in the course of mathematical analysis in the study of n-dimensional Euclidean space. Later, in the course of functional analysis, metric spaces are studied in more detail, in close connection with normalized and topological spaces. One of the obstacles in understanding the new facts of the metric spaces theory is the complexity of the geometric interpretation. For example, it is very difficult to imagine a single sphere in which many points are distant from each other at a constant distance. There are many similar examples in spaces with different metrics. They significantly prevent the adequate learning of the relevant properties of metric spaces. Moreover, with a change of space metric consisting of the same elements (points), the correlation between these elements can change significantly. For example, four points that are vertices of a square in two-dimensional arithmetic Euclidean space may become rectilinear and not flat placing if the metric of space will be changed. These examples are only a small part of the facts that can cause ambiguous students' perception of the metric spaces properties, and hinder the learning. On the other hand, the modern development of geometry (especially non-Euclidean geometries), the theory of infinitely measurable metric spaces, high technology, physics, cosmology indicate a significant use of metric spaces theory in practice and confirmation of its basic principles.

This paper proposes the geometrization methods of some metric spaces properties and their visualization. In our opinion, it will contribute to a deeper mastering of the basic concepts of the metric spaces theory by students, in particular, properties and correlations that have a geometric meaning. The main means of metric space geometrization is the concept of distance between each pair of its different points. We use this concept to formalize the concept of angle, forming three different points of this space. This approach makes it possible to consider the concept of rectilinear and flat placement of points of metric space. Moreover, it will allow their geometric interpretation and visualization in the usual Euclidean spaces. On the other hand, such an interpretation will allow a clear demonstration of cases of differences between individual geometric properties of metric spaces from Euclidean geometry.

The research can be largely attributed to the subject of metric geometry [1, 2], the rapid development of which in recent years is due to its significant applications in high technology, engineering and other fields of science and technology. The characteristic feature of metric geometry is that it is based only on the concept of distance between points and properties of a set of real numbers. It significantly limits the visualization of its results, but on the other hand, it expands and generalizes the classical concepts of Euclidean geometry. Metric geometry makes it possible to consider Euclidean geometry and non-Euclidean geometries. The studying method of the elements of metric geometry by students of higher education institutions, the use of applied graphical computer tools to illustrate the results of mathematical research were considered by a number of authors, in particular, visualization of basic concepts of spherical geometry was studied in [3, 4], the visualization of inequality solutions using the system of computer mathematics Maple was considered in [5], methodological aspects of the introduction of elements of metric geometry in the school course of mathematics were studied in [6, 7].

2. Preliminary information

The distance ϱ between the two elements x_i and x_j of the set X is called a real non-negative function $\varrho(x_i, x_j)$, which satisfies the condition of commutativity: $\varrho(x_i, x_j) = \varrho(x_j, x_i)$ and the condition of triangle inequality: $\varrho(x_i, x_j) \leq \varrho(x_i, x_k) + \varrho(x_k, x_j)$ for arbitrary points x_i, x_j, x_k of this space [8]. Such a set is called a metric space with metric ϱ, and is denoted by (X, ϱ).

Methods of distance introducing between points in space (metrization methods) can be varied [9]. Its geometric properties (space geometry) largely depend on the method of space metrization. In the following we will consider several classical spaces, the metrization is based on simple concepts, and which are easy to illustrate even on the material of the school course of mathematics [7].

- Set of ordered groups of n real numbers $(x_1, x_2, ..., x_n)$, where the distance between any two sets $x(x_1, x_2, ..., x_n)$ and $y(y_1, y_2, ..., y_n)$ is solved by the formula:

$$\varrho(x, y) = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2},$$

is a metric space, which is called n-dimensional arithmetic Euclidean space, and denote R^n [10, 11].

- Let's consider the set of continuous functions on the segment $[a, b]$. This set becomes a metric space [10, 11], if the distance between the two functions $f(t)$ and $g(t)$ of the set the number is taken:

$$\varrho(f, g) = max_{t\in[a,b]}|f(t) - g(t)|.$$

Such space is denoted by $C_{[a,b]}$.

- If on the set of continuous functions on the segment $[a, b]$ for the distance between the two functions $f(t)$ and $g(t)$ of the set the number is taken:

$$\varrho(f, g) = \int_a^b |f(t) - g(t)| dt,$$

then this set becomes a metric space, which is denoted by C_L [10].

Let's consider the concept of rectilinear placement of points of the metric space. For the convenience of further records we will use the notation: $\varrho(x_i, x_j) = \varrho_{ij}$ and considered that all points of the metric space are different, i.e. the value of the distance between them is always positive. Three points x_i, x_j, x_k of the metric space are placed rectilinearly in this space, if the equality is: $\varrho(x_i, x_k) = \varrho(x_i, x_j) + \varrho(x_j, x_k)$ [12] or shorter $\varrho_{ik} = \varrho_{ij} + \varrho_{jk}$. A set of points of a metric space will be called rectilinearly placed if every three points of this set are rectilinearly placed in this space.

Under the angle formed by three points x_i, x_j, x_k of the metric space, we understand the ordered trio of these points: (x_i, x_j, x_k) and denote $\angle(x_i, x_j, x_k)$, while the point x_j will be called the vertex of the angle and pairs of points (x_i, x_j) and (x_j, x_k) - its sides [13]. For the numerical characteristic φ of the angle (angular characteristic) it is natural to take the value of the cosine of the angle of the triangle, which is from the cosines formula in the Euclidean geometry [13,14]:

$$\varphi(x_i, x_j, x_k) = \frac{\varrho^2(x_i, x_j) + \varrho^2(x_j, x_k) - \varrho^2(x_i, x_k)}{2\varrho(x_i, x_j)\varrho(x_j, x_k)},$$

or shorter

$$\varphi_{ijk} = \frac{\varrho_{ij}^2 + \varrho_{jk}^2 - \varrho_{ik}^2}{2\varrho_{ij}\varrho_{jk}}. \quad (1)$$

Thus, the specified angular characteristic makes it possible not only to obtain the condition of rectilinear placement of three points of the metric space: $\varphi_{ijk}^2 = 1$, but also the condition of "lie between" of these points [15]. In particular, the point x_j "lies between" the points x_i and x_k (or is internal to the points x_i, x_j, x_k), if the equality: $\varphi_{ijk} = -1$, if the equality $\varphi_{ijk} = 1$, then we can say that the point x_j "lies outside" the points x_i and x_k (either is external or extreme for points x_i, x_j, x_k).

Using the angular characteristic, it is possible to determine the flat placement of the points of the metric space [13]. Four points x_1, x_2, x_3, x_4 of the metric space (X, ϱ) will be called flat placed in this space if the equality is:

$$1 + \varphi_{213}\varphi_{214}\varphi_{314} - \varphi_{213}^2 - \varphi_{214}^2 - \varphi_{314}^2 = 0. \quad (2)$$

The equality (2) in Euclidean geometry, in fact, means the equality of zero volume of a tetrahedron, its vertices are located at points x_1, x_2, x_3, x_4. If every four points of some set of metric space are flat placed in this space, this set will be called flat placed in this space.

Despite the similarity of the concepts of rectilinear and flat placing of points of metric space with the corresponding concepts of Euclidean geometry, they do not always coincide. In particular, the rectilinear placing of four points of a metric space does not always follow their flat placing in this space.

3. Geometric interpretation of particular metric geometry concepts

In contrast to Euclidean geometry, the geometric interpretation of the basic concepts of metric geometry causes some difficulties, because some facts that have no place in Euclidean geometry have to be depicted in two-dimensional or three-dimensional Euclidean spaces. Here are some examples of this interpretation.

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Example 1. On the segment $[0;1]$ let's consider four functions:

$$y_1 = 0, y_2 = 1, y_3 = x, y_4 = 1 - x.$$

Graphs of the functions y_1, y_2, y_3, y_4 can be represented on the coordinate plane in the space R^2 (Fig. 1).

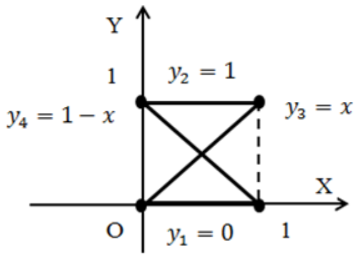

Figure 1. Images of functions y_1, y_2, y_3, y_4 in the coordinate system of space R^2.

If we consider the functions y_1, y_2, y_3, y_4 as points of the space $C_{[0;1]}$, then according to the metric of this space the distances between them will be:

$$\varrho_{12} = \varrho_{13} = \varrho_{14} = \varrho_{23} = \varrho_{24} = \varrho_{34} = 1.$$

In Euclidean geometry, such points form a regular tetrahedron with a single length of edges. Therefore, in order to interpret the mutual placing of these four points in the space $C_{[0;1]}$, it is necessary to move from their interpretation in the space R^2 to the space R^3. For convenience, we denote the points y_1, y_2, y_3, y_4, respectively, by the letters A, B, C, S. Choose a certain orientation of the tetrahedron in the space R^3. To do this, at representing the triangle ABC, point A will be placed on the coordinate start, point B- on the positive half of OX axis, point C - on the positive half of XOY plane, and point S - on the positive half of XYZ space (Fig. 2).

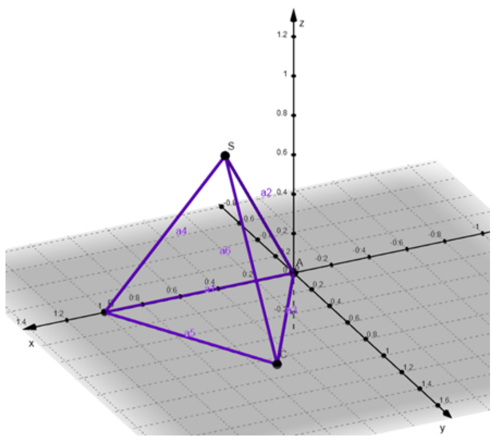

Figure 2. Location of points y_1, y_2, y_3, y_4 of space $C_{[0;1]}$ in the coordinate system of the space R^3.

This placing of points y_1, y_2, y_3, y_4 can be confirmed, using the definition of flat placing of points of metric space. To do this, we calculate by the formula (1) the angular characteristics for the angles that form the points y_1, y_2, y_3, y_4 in the space $C_{[0;1]}$:

$$\varphi_{213} = \varphi_{214} = \varphi_{314} = \frac{1^2 + 1^2 - 1^2}{2} = 0,5.$$

Putting these values into formula (2) we have:

$$1 + \varphi_{213}\varphi_{214}\varphi_{314} - \varphi_{213}^2 - \varphi_{214}^2 - \varphi_{314}^2 = 1 + (0,5)^3 - (0,5)^2 - (0,5)^2 - (0,5)^2 \neq 0.$$

Therefore, the points y_1, y_2, y_3, y_4 are not flat placing in the space $C_{[0;1]}$.

The image in Figure 2 was obtained using the dynamic geometric environment GeoGebra 3D. In this environment, it is possible visually make sure that the points y_1, y_2, y_3, y_4 are not really flat placing by rotating the coordinate system. The coordinate system is rotated so that the points A, B, C, S can be observed from a point on the XOY flat (Fig. 3).

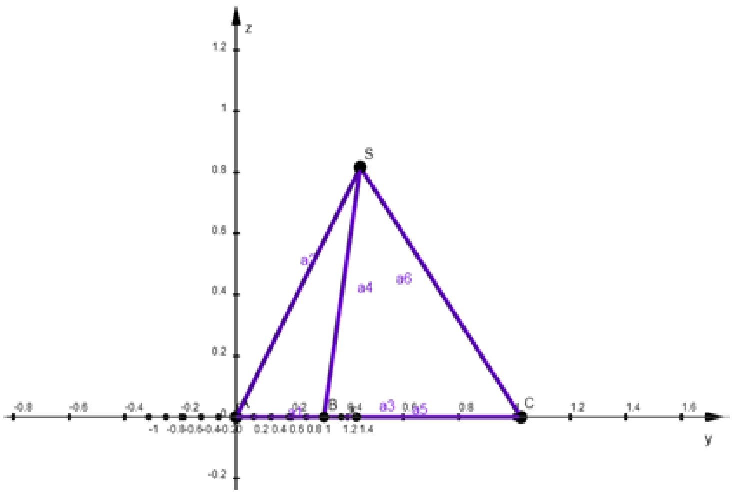

Figure 3. Placement of points y_1, y_2, y_3, y_4 when rotating the coordinate system.

If we consider the functions y_1, y_2, y_3, y_4 as points of the space C_L, according to the metrics of this space, the distances between them will be:

$$\varrho_{12} = 1; \varrho_{13} = \varrho_{14} = \varrho_{23} = \varrho_{24} = \varrho_{34} = 0,5.$$

At such distances between these points, their mutual placing cannot be represented in Euclidean geometry. They cannot lie on one line (points y_1, y_3, y_4 form an equilateral triangle). These points cannot lie in the same flate, because the points y_1, y_2, y_3, are placed rectilinearly ($\varrho_{13} + \varrho_{23} = \varrho_{12}$), and the points y_1, y_2, y_4 are placed rectilinearly ($\varrho_{14} + \varrho_{24} = \varrho_{12}$), so the points y_3 and y_4 coincide, although the distance between them is positive. In addition, a tetrahedron cannot be constructed from the points y_1, y_2, y_3, y_4, because in any orientation its three vertices will be rectilinear placed.

It is possible to construct an interpretation of such placing using a sphere in the space R^3. The points y_1, y_2, y_3, y_4 will be placed on the hemisphere of radius $\dfrac{1}{\pi}$. Let's take the length of the arc of the great semicircle that connects these points for the distance between a pair of points on the hemisphere (Fig. 4).

In this interpretation, the points y_1 and y_2 will be the ends of the large circle diameter, its length is two units. The points y_3 and y_4 will lie in the middle of the two semicircles connecting

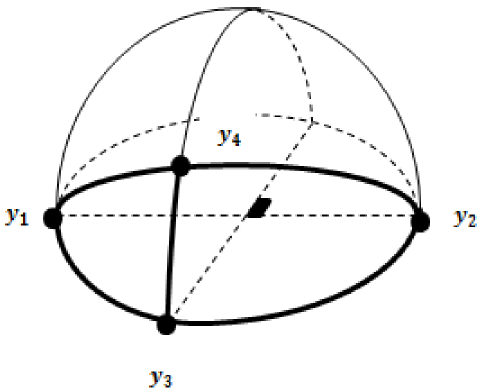

Figure 4. Interpretation of the location of points y_1, y_2, y_3, y_4 of the space C_L in the space R^3.

the points y_1 and y_2. Moreover, the points y_1, y_3, y_4, as well as points y_2, y_3, y_4, will form equilateral spherical triangles.

Example 2. On the segment $[0;1]$ consider the following functions: $y_1 = x$, $y_2 = -x$, $y_3 = -x+1$, $y_4 = x-1$ (Fig. 5).

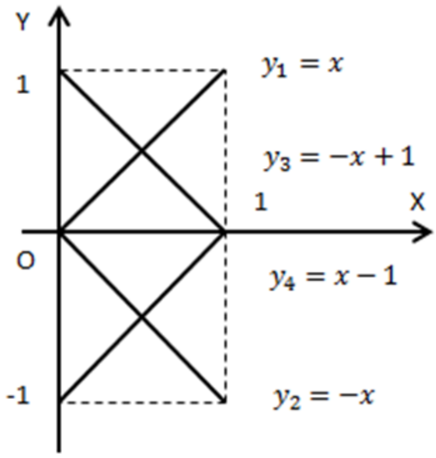

Figure 5. Images of functions y_1, y_2, y_3, y_4 in the coordinate system of space R^2.

The study [7] shows the functions y_1, y_2, y_3, y_4, as points of the space $C_{[0;1]}$, are rectilinearly placing in this space. Its peculiarity is each of these points "lies between" some of them. In the Euclidean geometry, among the four rectilinear points, two of them will necessarily be "extreme" and two – "internal" [16]. Moreover, unlike the Euclidean geometry, the points y_1, y_2, y_3, y_4 will not be flat placing in the space. Indeed, we find the distances between these points by the metric of space $C_{[0;1]}$:

$$\varrho_{12} = 2; \varrho_{13} = \varrho_{14} = \varrho_{23} = \varrho_{24} = 1; \varrho_{34} = 2.$$

Now, by the formula (1) we find the numerical characteristics of angles that have a vertex, such as point y_1:

$$\varphi_{213} = \frac{\varrho_{12}^2 + \varrho_{13}^2 - \varrho_{23}^2}{2\varrho_{12}\varrho_{13}} = \frac{2^2 + 1^2 - 1^2}{4} = 1;$$

$$\varphi_{214} = \frac{\varrho_{12}^2 + \varrho_{14}^2 - \varrho_{24}^2}{2\varrho_{12}\varrho_{14}} = \frac{2^2 + 1^2 - 1^2}{4} = 1;$$

$$\varphi_{314} = \frac{\varrho_{13}^2 + \varrho_{14}^2 - \varrho_{34}^2}{2\varrho_{13}\varrho_{14}} = \frac{1^2 + 1^2 - 2^2}{2} = -1.$$

Substitute these values in the left part of formula (2):

$$1 + \varphi_{213}\varphi_{214}\varphi_{314} - \varphi_{213}^2 - \varphi_{214}^2 - \varphi_{314}^2 = 1 + (-1) - 1^2 - 1^2 - (-1)^2 \neq 0.$$

Therefore, the points y_1, y_2, y_3, y_4 are not flat placing in the space $C_{[0;1]}$.

If students are asked to give a geometric interpretation of Example 2 in the space R^2, it can cause difficulties, because in this space a straight line always belongs to the flate. However, such an interpretation is possible if the distance between the points in the space $C_{[0;1]}$ is the length of some arc of the line connecting these points. For example, to show the mutual placing of points y_1, y_2, y_3, y_4 in the space R^2, we place them on a circle of radius $\frac{2}{\pi}$, and for the distance between a pair of these points we can take the length of the smaller of the two arcs connecting these points (Fig. 6).

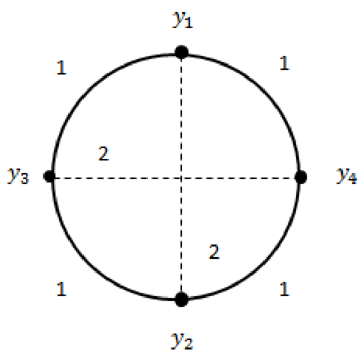

Figure 6. Interpretation of the rectilinear placement of points y_1, y_2, y_3, y_4 of the space $C_{[0;1]}$ in the space R^2.

The fact the points y_1, y_2, y_3, y_4 are not flat placing in the space $C_{[0;1]}$ can be conveniently illustrated, as in Example 1, on the hemisphere of radius $\frac{2}{\pi}$ in the space R^3 (Fig. 7).

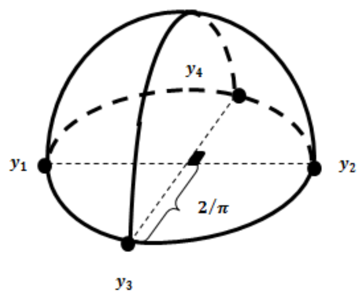

Figure 7. Interpretation of the location of points y_1, y_2, y_3, y_4 of the space $C_{[0;1]}$ in the space R^3.

In this interpretation, the points y_1, y_2, y_3, y_4 will be the ends of two mutually perpendicular diameters of a large circle. Its length is four units.

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4. Conclusions

The examples of geometric interpretation and visualization of the mutual points placing of metric space given in this paper can contribute to a deeper and more conscious perception and understanding of the metric spaces theory. The analogy of particular connections between the points of metric space with the corresponding connections in Euclidean geometry makes it possible to trace the change in the characteristic geometric properties of space at its metric changes. The use of special graphical capabilities of the corresponding software allows not only to visualize the mutual points placing of the metric space, but also to track its change at changing observation point of placing.

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