Potential accuracy of speckle-interferometric measurements. Angular diameters and limb darkening of stars

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The potentialities of the joint estimation of several unknown parameters of an arbitrary object from the results of measuring the power spectrum of speckle images are investigated. A typical problem of the joint estimation of the angular diameter and the coefficient of limb darkening of a stellar disk with a linear darkening law is analyzed. It is shown that strongly correlated estimates of the unknown parameters are obtained in this case, regardless of the means of estimation, and the correlation coefficient of the jointly effective (Kramer–Rao) estimates is a function of the observation circumstances and the characteristics of the observed object. An optimum strategy for solving this problem is proposed, allowing for the invariance of the average number of detected photons per speckle for different models describing the brightness distribution over a stellar disk. The results obtained make it possible to plan an experiment to solve the problem of measuring the angular diameter and limb darkening of a star by the method of speckle interferometry.

INTRODUCTION

In the first paper of this work[1] a general relation was obtained determining the lower limit (the Kramer–Rao limit) of the dispersion of an estimate of an unknown scalar parameter p of an arbitrary object from the results of measuring the power spectrum of speckle images:

$$\sigma_p^2 = \frac{1}{2(M-1)K_p}, \quad (1)$$

where

$$K_p = n_{os}^2 \int_0^{\pi} \int_0^{\nu_D} \frac{[g_D(\nu)]^2 \left[\frac{\partial \Phi(\nu, \psi, p)}{\partial p}\right]^2 \nu \, d\nu \, d\psi}{[1+n_{os} g_D(\nu) \Phi(\nu, \psi, p)]^2}, \quad (2)$$

n_{os} is the average number of detected photons per speckle, M is the number of speckle images, $\Phi(\nu, \psi, p) = |f_n(\nu, \psi, p)|^2$ is the normalized square of the absolute value of the Fourier transform of the brightness distribution over the object, $g_D(\nu)$ is the diffractional optical transmission function (OTF) of the telescope, and ν_D is the diffraction-limited spatial frequency in the telescope. The problem of measuring the angular diameter of a star for the model of a uniformly luminous (uniform) disk was investigated using this relation.

It is obvious that an angular diameter obtained under the assumption of a uniform disk has a somewhat arbitrary character. It is also obvious that the more perfected the model representing the brightness distribution over the stellar disk, the closer the angular diameter will be to the true value. The relative brightness distribution over the projection of the visible surface of a star onto the picutre plane is expressed by the limb-darkening law. Ideally, of course, one would wish to obtain not only the angular diameter but also the brightness distribution over the disk for the measurements, since this information is just what makes it possible to find the source function in the stellar atmosphere. In the final analysis, this would be a powerful means of testing theoretical models of stellar atmospheres.

For an extensive class of models of thin stellar atmospheres, the theory of Ref. 2 yields a so-called linear law of limb darkening,

$$I(\chi) = I(0)[1-u(1-\cos\chi)], \quad (3)$$

where I(0) is the brightness at the center of the stellar disk, χ is the angle between the line of sight and the normal to the stellar surface, and u is the coefficient of limb darkening.

The stars currently accessible to the method of speckle interferometry, however, evidently possess extended atmospheres,[3] and the linear law (3) is inapplicable to them, strictly speaking. On the other hand, at the present time there is no satisfactory theory of the escaping radiation for extended atmospheres,[4,5] and expressions describing limb darkening in analytic form do not exist. Therefore, the linear law (3) is used, as a rule, to interpret interferometric observations[6,7] or the light curves of eclipsing binary systems.[8]

Investigations of the influence of effects of limb darkening on the measurement of angular diameters by intensity interferometry were made by Hanbury Brown et al.[6] Besides (3), they included in the analysis the more complicated law

$$I(\chi) = I(0)[1-u(1-\cos\chi)-v(1-\cos\chi)^2-w(1-\cos\chi)^3]. \quad (4)$$

This analysis showed that the correlation curve of intensity fluctuations, measured by a given interferometer and proportional, as is well known, to the coherence function $\Gamma^2(d)$ of the source, where d is the interferometer base line, in its shape up to the first minimum is not sensitive to the brightness distribution over the stellar dark. Moreover, they concluded that if the measurements of the coherence function of the source cover it only up to the second minimum, it is practically impossible to distinguish between different darkening laws, e.g., (3) and (4), or to give preference to some choice from the set of values of u, v, and w in (4). Therefore, observational data were interpreted at first through the angular diameter θ_{UD} of the equivalent uniform disk. Then to obtain the true angular diameter θ_{LD} of the limb-darkened stellar disk, a correction was introduced through the approximate formula

$$\frac{\theta_{LD}}{\theta_{UD}} = \left[\frac{1-u/3}{1-7u/15}\right]^{1/2}, \quad (5)$$

where the values of u were assigned by fitting Eq. (3), by the method of least squares, to the brightness distribution over the disk given by a certain model stellar atmosphere. Equation (5) approximately (to within 1%, however) reflects the connection between θ_{LD} and θ_{UD} obtained through the appropriate scaling with respect to the argument of the coherence functions of the source, under the condition that they coincide at the point $\Gamma^2(d) = 0.3$. This point was chosen because the majority of measurements of angular diameters by an intensity interferometer were made for $\Gamma^2(d) = 0.3$. The use of Eq. (4) instead of (3) hardly alters the final value of θ_{LD}.

STATEMENT OF THE PROBLEM

A question arises: Is it possible to determine the angular size of a star and the brightness distribution over its disk, described by a limb-darkening law, from the power spectrum of speckle images? Of course, in the given context the problem must have the following statement. A suitable limb-darkening model, characterized by one or several parameters, is chosen from a priori considerations. It is required to find out how precisely one can estimate this or these parameters jointly with the estimation of the angular diameter by the method of speckle interferometry. The problem can be formulated differently: With what confidence can once decide, within the framework of a given method of data collection and treatment, which of a set of possible models best describes the object?

As was done in Ref. 1, we represent the result of measuring the quantity $\Phi(\mathbf{v}, \mathbf{p})$ at the i-th point of the power spectrum in the form

$$S_i = \Phi(\mathbf{v}_i, \mathbf{p}) + v_i, \quad i = 1, 2, \ldots, k, \tag{6}$$

where $\mathbf{p} = \mathbf{p}(p_1, p_2, \ldots, p_N)$ is the vector parameter to be estimated, k is the total number of measurements, equal to the number of independent reading points in the power spectrum, and v_i are the independent measurement errors, with a zero mathematical expectation [assuming an adequate model $\Phi(\mathbf{v}, \mathbf{p})$) and a dispersion

$$\sigma_i^2 = \frac{[1 + n_{os} g_D(v_i) \Phi(\mathbf{v}_i, \mathbf{p})]^2}{n_{os}^2 (M-1) [g_D(v_i)]^2}. \tag{7}$$

Equation (7) allows for the influence of speckle noise and quantum fluctuations and is valid under the conditions[9] $n_T > 1$ and $(D/r_0)^2 \gg 1$, where n_T is the average number of photons detectable in one speckle image, D is the telescope diameter, and r_0 is the coherence radius of the wave front, distorted by atmospheric turbulence, in the aperture (Fried's parameter[10]).

GENERAL SOLUTION

Let $G(\mathbf{p})$ be the covariation matirx of the estimate of the vector $\mathbf{p}$ (the matrix of errors) and A be the matrix of partial derivatives $\partial\Phi(\mathbf{v}, \mathbf{p})/\partial p_j$ (j = 1, 2, ..., N), which exist for all $\mathbf{p}$; Ω is a secular matrix with diagonal elements $\omega_i = 1/\sigma^2_i$ (i = 1, 2, ..., k), where σ^2_i are defined by Eq. (7). We can show that the matrix $G(\mathbf{p}) - (A^t \Omega A)^{-1}$ is nonnegative definite for all $\mathbf{p}$, which can be written as

$$G(\mathbf{p}) \geqslant (A^t \Omega A)^{-1}, \tag{8}$$

where t is the transposition symbol. The inequality (8) is a generalization of the Kramer–Rao inequality to the case of a vector parameter.[11] The lower limit in (8) is reached if, first, the measurement errors [i.e., the values of v_i in (6)] are distributed normally, which will occur for a sufficiently large volume M of the sample of speckle images by virtue of the central-limit theorem. Second, the function $\Phi(\mathbf{v}, \mathbf{p})$ must be linear with respect to $\mathbf{p}$. If $\Phi(\mathbf{v}, \mathbf{p})$ is a nonlinear function, then the estimate $\hat{\mathbf{p}}$ will be effective only with an unlimited increase in the number of mesaurements $(k \to \infty)$,[12] i.e., asymptotically. The number of independent reading points in the power spectrum is[1] $k \propto (D/r_0)^2$, and for $(D/r_0)^2 \gg N$ one can obtain an estimate $\hat{\mathbf{p}}$, sufficiently close to the effective value if $\Phi(\mathbf{v}, \mathbf{p})$ is a nonlinear function.

The diagonal elements of the covariation matrix $G(\mathbf{p})$ represent the dispersions $\sigma_1^{*2}, \sigma_2^{*2}, \ldots, \sigma_N^{*2}$ of estimates of the components of the parametric vector $\mathbf{p}$, while the nondiagonal elements represent $\mathrm{Cov}(p_j, p_\ell) = q_{j\ell}\sigma^*_j\sigma^*_\ell$, where $q_{j\ell} = q(p_j, p_\ell)$ is the correlation coefficient of estimates of the components p_j and p_ℓ.

The correlation coefficient of the jointly effective estimates $\hat{p}_j$ and $\hat{p}_\ell$, according to (8) and with allowance for (7), is

$$q_{jl} = \frac{K_{jl}}{\sqrt{K_j}\sqrt{K_l}}, \tag{9}$$

where

$$K_{jl} = \int_0^\pi \int_0^{v_D} \frac{[g_D(v)]^2 \left(\frac{\partial\Phi}{\partial p_j}\right)\left(\frac{\partial\Phi}{\partial p_l}\right) v\, dv\, d\psi}{[1 + n_{os} g_D(v) \Phi(v, \psi, \mathbf{p})]^2}, \tag{10}$$

$K_j = K_{jj}$, and $K_\ell = K_{\ell\ell}$, i.e., K_j and K_ℓ have the same meaning as in (2) only without the quantity n_{os}^2 in front of the integral. Here, as in Ref. 1, the transition is made from summation over the independent reading points to integration over the entire power specturm.

For signals with low light levels, i.e., for $n_{os} \ll 1$, the term $n_{os} g_D(v) \Phi(v, \psi, \mathbf{p})$ in the denominator of the integrand of (10) can be neglected. Then

$$K_{jl} = \int_0^\pi \int_0^{v_D} [g_D(v)]^2 \left(\frac{\partial\Phi}{\partial p_j}\right)\left(\frac{\partial\Phi}{\partial p_l}\right) v\, dv\, d\psi, \tag{11}$$

and the integrals K_j and K_ℓ are simplified similarly. Thus, in this case the correlation coefficient $q_{j\ell}$ ceases to depend on n_{os}.

When the components $p_1, p_2, \ldots, p_N$ are estimated separately (when all but one of them are

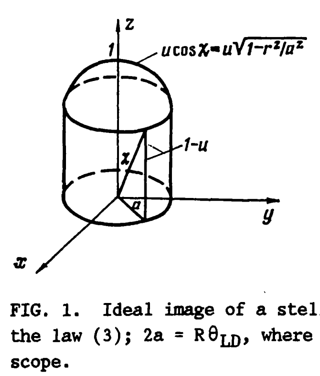

FIG. 1. Ideal image of a stellar disk with limb darkening by the law (3); $2a = R\theta_{LD}$, where R is the focal length of the telescope.

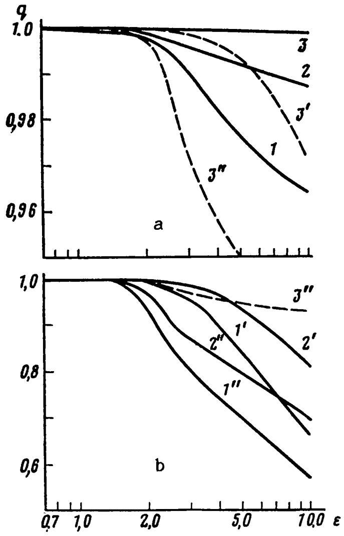

FIG. 2. Correction coefficient q of jointly effective estimates of the parameters u and ε as a function of $\varepsilon = \theta_{LD}D/\lambda$: 1, 1', and 1") u = 0; 2, 2') and 2") u = 0.5; 3, 3', and 3") u = 1 for $n_{os} \ll 1$, $n_s = 1$, and $n_s = 10^2$, respectively. Region of values of q: a) $0.95 \leq q \leq 1$; b) $0.5 \leq q \leq 1$.

assumed to be known), the accurcy of the effective estimate is determined by Eqs. (1) and (2). In a joint estimate, when N = 2, the Kramer–Rao limit for σ_1^{*2} and σ_2^{*2} is determined by the relations[13]

$$\sigma_1^{*2} = \frac{\sigma_1^2}{1-q^2}, \quad \sigma_2^{*2} = \frac{\sigma_2^2}{1-q^2}, \tag{12}$$

where σ_1^2 and σ_2^2 are the dispersions of the separately effective estimates $\hat{p}_1$ and $\hat{p}_2$, respectively, and $q = q(p_1, p_2)$ is defined by Eq. (9).

CORRELATION COEFFICIENT OF ESTIAMTES FOR A LINEAR DARKENING LAW

Let us analyze the possibility of a joint estimate of the angular diameter θ_{LD} and the coefficient of limb darkening u from the power spectrum, under the assumption that the brightness distribution over the stellar disk is described by Eq. (3). We find the Hankel transformation of $I(\chi)$ (Fig. 1) using the integral[14]

$$\int_0^1 x^{\alpha+1}(1-x^2)^\beta J_\alpha(bx)\,dx = 2^\beta\Gamma(\beta+1)\,b^{-(\beta+1)}J_{\alpha+\beta+1}(b), \tag{13}$$

$$[b>0,\ \operatorname{Re}\alpha>-1,\ \operatorname{Re}\beta>-1].$$

In our case $\alpha = 0$ and $\beta = 1/2$. As a result, the normalized square of the absolute value of the spectrum of the object has the form

$$\Phi(\xi) = \frac{9}{(3-u)^2}\left[(1-u)\frac{2J_1(\pi\varepsilon\xi)}{\pi\varepsilon\xi} + u\sqrt{2\pi}\frac{J_{3/2}(\pi\varepsilon\xi)}{(\pi\varepsilon\xi)^{3/2}}\right]^2, \tag{14}$$

where $\varepsilon = \theta_{LD}D/\lambda$ is the measure of the closeness of the limb-darkened disk to a diffraction element of resolution of the telescope of diameter D; $\xi = \nu/\nu_D$ $(0 \leq \xi \leq 1)$, where $\nu_D = D/\lambda$ is the diffraction-limited frequency in the telescope; J_1 and $J_{3/2}$ are Bessel functions of order 1 and $3/2$, respectively.

The adopted model (3) of limb darkening, for an average number n_{os} of photographic events per speckle, yields

$$n_{os} = n_s\left[1 + \left(1 - \frac{u}{3}\right)\varepsilon^2\right], \tag{15}$$

where n_s is the average number of photographic events per diffraction element of resolution, which we define as[1]

$$n_s = 0{,}342\eta n_V t_E \Delta\lambda r_0^2. \tag{16}$$

Here n_V is the spectral flux density from the star [quanta/(cm^2·sec·nm)], t_E is the time of exposure of one speckle image, and $\Delta\lambda$ is the half-width of the spectral band of the filter. By the quantity η we understand the equivalent quantum yield (DQE of Refs. 15 and 16) of the speckle camera.

We calculate dthe correlation coefficients of estimates of the parameters u and ε for several values of u and n_s on Nairi-K and ES-1022 computers in accordance with (9). For $g_D(\xi)$ we used the OTF of a telescope with a continuous round aperture,

$$g_D(\xi) = \frac{2}{\pi}(\arccos\xi - \xi\sqrt{1-\xi^2}). \tag{17}$$

In Fig. 2 we give the dependence of q on ε for fixed values of u of 0, 0.5, and 1 for the cases of $n_{os} \ll 1$, $n_s = 1$, and $n_s = 10^2$. It is seen from Fig. 2 that, first, the correlation coefficient decreases with an increase in n_{os} while it increases with an increase in u; second, in the case of $n_{os} \ll 1$ the correlation coefficient differs little from one for all values of ε up to ε = 10. The same thing is observed at $n_{os} \geq 1$ for darkening coefficients u > 0.5. The presence of such a strong correlation between the quantities u and ε estimated from the power spectrum means that it is very difficult, in practice, to separate their influence on the shape (profile) of the spectrum, while for ε < 1.5 it is, perhaps, fundamentally impossible to do this.

We emphasize that the correlation coefficient that we have found is not selective and reflects a fundamental property of the estimation (regardless of the method) of the parameters u and ε on the basis of measurements of the power spectrum of speckle images. It is obvious that the quantity q reflects the linear component of the spurious functional dependence arising within the framework of the given method of data collection and treatment.

From Eqs. (12) it follows that as $q \to 1$, no matter what σ_u and σ_ε (the rms errors of the separate estimates) are, the values of σ^*_u and σ^*_ε grow without limit, and a joint estimate of the parameters losses any meaning. Even for large ε and (or) n_s, excluding the case of $u \leq 0.5$, the situation remains extremely poor. And it is not only that the values of σ^*_u and σ^*_ε will be large. For estimation, e.g., by the method of least squares, the matrix of coefficients of the system of normal equations will be poorly conditional – a rather typical case in problems of nonlinear estimation. Specially developed methods of improving the conditionality of the matrix exist for the solution of such problems (the Marquardt and Jones methods, methods of "choice of directions," etc.[17]), but even they do not always lead to succes. In any case, the indeterminacy inherent to the estimate $\hat{u}$ and $\hat{\varepsilon}$ remains. An error, e.g., in the estimate $\hat{u}$ must lead to an error in the estimate $\hat{\varepsilon}$ and vice versa, i.e., the soltuion cannot be considered stable

COUPLING EQUATION

One circumstance exists which must always be kept in mind. No matter what model brighness distribution over the stellar disk is proposed, one thing remains unchanged: the average number of photographic events per speckle for the given circumstances of observation. In the first article [1] of this work it was shown that the model of a uniform disk gives

$$n_{0s}=n_s(1+\varepsilon_0^2), \tag{18}$$

where $\varepsilon_0 = \theta_{UD}D/\lambda$ is the measure of the closeness to a uniform disk. Equating the right sides of Eqs. (15) and (18) to each other, we find

$$\frac{\varepsilon}{\varepsilon_0}=\frac{\theta_{LD}}{\theta_{UD}}=\left[\frac{3}{3-u}\right]^{1/2}. \tag{19}$$

This relation gives barely larger corrections than the relation (5) in the transition from θ_{UD} to θ_{LD} for a known u. In the transition from θ_{UD} to θ_{LD} for a fully darkened disk (u = 1), e.g., a correction factor 1.118 is found from Eq. (5), while 1.225 is found from Eq. (19). Remember that Eq. (5) is approximate; the exact value of the correction factor according to Hanbury Brown et al.[6] is 1.127, although they ultimately corrected their values of the angular diameters (for the case of u = 1) with a coefficeint 1.134.

With allowance for the foregoing, the following course seems natural in a joint estimate of u and ε. First we estimate the value of ε_0, the measure of closeness to the equivalent uniform disk. This can be done, e.g., by the method of least squares (MLS): In nonlinear estimation, MLS estimates are asymptotically effective.[17] Then we find the MLS estimates of the parameters u and ε, using the function (14) when the coupling equation (19) holds between the parameters. This can be done, e.g., by the method of Lagrangian multipliers.[18]

First of all, however, we must answer a question: Does such an estimation procedure always make sense? Can it be that the difference between these models (of a uniform and a limb-darkened disk), under the condition that their parameters are connected by Eq. (19), is so slight that it is easily smoothed over by measurement errors? Let us try to find out.

PROBABILITY OF A CORRECT DECISION

To simplify the discussions, we agree to assume that the models under consideration adequately describe the corresponding objects. An answer to the question of which of two possible objects yielded the given power spectrum can be obtained by methods of the statistical theory of decision making (or the theory of verification of hypotheses). Since probability-ratio criteria are the most powerful of all possible criteria (the Neymann–Pearson lemma),[19] as the critical statistic we shall use the logarithm of the probability ratio (the criterion was first used to distinguish images of Harris[20]).

For convenience, we designate $\Phi(\xi_i, \varepsilon_0) = \Phi_{1i}$ a and $\Phi(\xi_i, u, \varepsilon) = \Phi_{2i}$, where the index i takes values from 1 to k, while k is the number of independent reading points in the power spectrum, as before. As we did above, we represent the result of the measurement of the quantity Φ at the i-th point in the form (6). Then the probability function due to the presence of Φ_1 has the form

$$L(\Phi_1)=\prod_{i=1}^{k}\left(\frac{\omega_i}{2\pi}\right)^{1/2}\exp\left[-\frac{\omega_i(S_i-\Phi_{1i})^2}{2}\right], \tag{20}$$

where we take the average error of unit weight as one. Similarly,

$$L(\Phi_2)=\prod_{i=1}^{k}\left(\frac{\omega_i}{2\pi}\right)^{1/2}\exp\left[-\frac{\omega_i(S_i-\Phi_{2i})^2}{2}\right]. \tag{21}$$

If the alternatives Φ_1 and Φ_2 are a priori equally probable, then the descision must be made on the principle of maximum probability. We define the critical statistic as

$$\gamma_{12}=2\ln\frac{L(\Phi_1)}{L(\Phi_2)}=\sum_{i=1}^{k}[\omega_i(S_i-\Phi_{2i})^2-\omega_i(S_i-\Phi_{1i})^2]. \tag{22}$$

Obviously, if $\gamma_{12} > 0$, the decision should be made in favor of Φ_1; otherwise, i.e., for $\gamma_{12} < 0$, the decision should be made in favor of Φ_2. Knowing the distribution of the quantity γ_{12}, one can calculate the probability of making the correct decision.

In fact, suppose that we have analyzed an object having a spectral square Φ_1. Then, substituting $S_i = \Phi_{1i} = v_i$ into (22), we obtain

$$\gamma_{12}(\Phi_1)=\sum_{i=1}^{k}\omega_i[(\Phi_{1i}-\Phi_{2i})^2+2v_i(\Phi_{1i}-\Phi_{2i})]. \tag{23}$$

Since the quantity v_i is normally distributed, $\gamma_{12}\cdot(\Phi_1)$ also has a normal distribution with a mean value

$$\mu=\langle\gamma_{12}\rangle=\sum_{i=1}^{k}\omega_i(\Phi_{1i}-\Phi_{2i})^2 \tag{24}$$

and a dispersion defined in the well-known way: $\sigma_\gamma^2 = \langle\gamma_{12}^2\rangle - \mu^2$. It is easy to show that

$$\sigma_\gamma^2=4\sum_{i=1}^{k}\omega_i(\Phi_{1i}-\Phi_{2i})^2. \tag{25}$$

The probability of a correct decision is calculated as follows:

$$P=\frac{1}{\sigma_\gamma\sqrt{2\pi}}\int_0^\infty\exp\left[-\frac{(\gamma_{12}-\mu)^2}{2\sigma_\gamma^2}\right]d\gamma_{12}. \tag{26}$$

Performing the change of variables $(\gamma_{12} - \mu)/\sigma_\gamma = z$, we obtain

$$P=\frac{1}{\sqrt{2\pi}}\int_{-\mu/\sigma_\gamma}^{\infty}\exp\left(-\frac{z^2}{2}\right)dz, \tag{27}$$

where

$$\frac{\mu}{\sigma_\gamma}=\frac{1}{2}\left[\sum_{i=1}^{k}\omega_i(\Phi_{1i}-\Phi_{2i})^2\right]^{1/2}. \tag{28}$$

Converting to integral form and using (7), we find

$$\frac{\mu}{\sigma_\gamma}=\frac{1}{2}\sqrt{(M-1)\pi K_\Phi}, \tag{29}$$

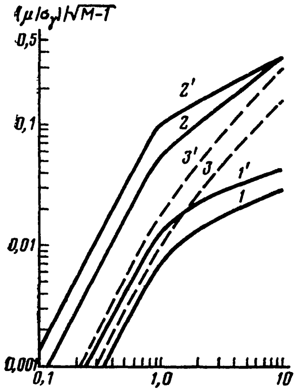

FIG. 3. Dependence of the quantity μ/σ_γ, normalized so that it does not depend on M, on the measure of closeness to an equivalent uniform disk, $\varepsilon_0 = \theta_{UD}D/\lambda$: 1 and 1') $n_s = 1$; 2 and 2') $n_s = 10^2$ for u = 0.5 and 1, respectively. Dashed lines denote the dependence of $(\mu/\sigma_\gamma)/n_s\sqrt{M-1}$ on ε_0 in the case of $n_s(1+\varepsilon_0^2) \ll 1$: 3) u = 0.5; 3') u = 1.

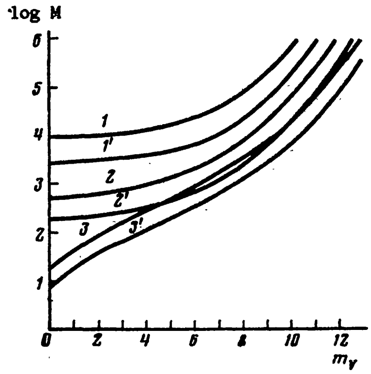

FIG. 4. Number M of speckle images required to make the correct decision with a probability P = 0.99 as a function of the stellar magnitude m_V of the object: a and 1') $\varepsilon_0 = 0.5$; 2 and 2') $\varepsilon_0 = 1$; 3 and 3') $\varepsilon_0 = 2$ for u = 0.5 and 1, respectively. Observing conditions: $r_0 = 10$ cm, $t_E = 0.01$ sec, $\Delta\lambda = 40$ nm, $\eta = 10\%$.

where

$$K_\Phi = \int_0^1 \frac{[n_s(1+\varepsilon_0^2)g_D(\xi)\delta\Phi(\xi)]^2\xi\, d\xi}{[1+n_s(1+\varepsilon_0^2)g_D(\xi)\Phi_1(\xi)]^2}, \qquad (30)$$

$$\delta\Phi(\xi) = \Phi_1(\xi,\varepsilon_0) - \Phi_2(\xi,u,\varepsilon) = \left[\frac{2J_1(\pi\varepsilon_0\xi)}{\pi\varepsilon_0\xi}\right]^2 - \frac{9}{(3-u)^2}\left[(1-u)\frac{2J_1(\pi\alpha\varepsilon_0\xi)}{\pi\alpha\varepsilon_0\xi} + u\sqrt{2\pi}\frac{J_{3/2}(\pi\alpha\varepsilon_0\xi)}{(\pi\alpha\varepsilon_0\xi)^{3/2}}\right]^2, \qquad (31)$$

and $\alpha = \varepsilon/\varepsilon_0$ and is defined by Eq. (19).

For signals with low light levels, i.e., for $n_s(1+\varepsilon_0^2) \ll 1$, we have

$$\frac{\mu}{\sigma_\gamma} = \frac{1}{2} n_s(1+\varepsilon_0^2)\sqrt{(M-1)\pi K_\Phi}, \qquad (32)$$

where

$$K_\Phi = \int_0^1 [g_D(\xi)]^2[\delta\Phi(\xi)]^2\xi\, d\xi. \qquad (33)$$

Thus, the probability of a correct decision is a function of the circumstances of observation and the characteristics of the object observed.

In Fig. 3 we give the dependence of $(\mu/\sigma_\gamma)\sqrt{M-1}$ on ε_0 for the values of u = 0.5 and 1 and $n_s = 1$ and 10^2. The dashed lines in this figure correspond to the dependence for $(\mu/\sigma_\gamma)/n_s\ (M-1)^{\frac{1}{2}}$ in the case of $n_s(1+\varepsilon_0^2) \ll 1$.

Let us determine the number M of speckle images required to make the correct decision with a probabiltiy P = 0.99 (i.e., the probability of a false alarm in 0.01). One should not be misled by such a high probability, since the value P = 0.5 corresponds to random decision making under the conditions of an alternative. Using normal-distribution tables,[18] we find that for P = 0.99 the value of μ/σ_γ should be 2.33. From (29) we obtain

$$M = 1 + \frac{6.91}{K_\Phi}. \qquad (34)$$

In Fig. 4 we show the dependence of the necessary number M of speckle images on the stellar magnitude m_V of the object when $\varepsilon_0 = 0.5$, 1, and 2 for u = 0.5 and 1. The quantity m_V appears in Eq. (34) through Eq.(16) by way of the well-known relative

$$n_V = n_0 10^{-0.4 m_V}, \qquad (35)$$

where n_0 is the spectral flux density from a zeroth-magnitude star, equal to 10^4 quanta/(cm^2·sec·nm) in the green region of the spectrum.[21] The following observing conditions were assumed: $\eta = 10\%$, $t_E = 0.01$ sec, $\Delta\lambda = 40$ nm, and $r_0 = 10$ cm. Here we did not allow for light losses in the atmosphere and in the optics of the telescope and the speckle camera.

Thus, the proposed strategy makes it possible in each concrete situation to judge the correctness of a joint estimate of the unknown parameters u and ε. For this it is sufficient to know the measure of the closeness to the equivalent uniform disk, since the dependence of μ/σ_γ on u is rather weak, as can be seen from Fig. 3. Consequently, one should always start with the estimate of ε_0. And if it turns out that the brightness of the object and the circumstances of observation (i.e., the values of η, r_0, and M) do not allow one to make a correct decision, with a sufficiently high probability, about the adequacy of the model ascribed to the given object, then one ends up being confined to the simplest model – the model of an equivalent uniform disk. Otherwise (when there is sufficient information in the sample of speckle images), one can change to a more complicated model and seek a joint estimate of the angular diameter and the coefficient of limb darkening in the presence of a coupling equation, which is a consequence of the invariance of the average number of photographic events per speckle for different models describing the brightness distribution over the stellar disk.

CONCLUDING REMARKS

1. Speckle-interferometric measurements of the angular diameter and limb darkening [by the law(3)] of the supergiant α Ori, made by Wilkerson and Worden [7] using a 4-m telescope, yielded the following results: in the continuum (510 ± 5 nm region) $\theta_{LD} = 0''.0520 \pm 0''.0017$ and u = 0.75 ± 0.13 and in the TiO band (520 ± 5 nm) $\theta_{LD} = 0''.0569 \pm 0''.0010$ and u = 0.93 ± 0.03. The high accuracy of the estiamtes of θ_{LD} and u, especially in the TiO band, obtained from 180 speckle images with the best, it is stated, agreement of the data, attracts attention. And this is for $\varepsilon_0 \approx 1.7$. Moreover, the discovered effect of an increase in dark-

ening in the transition from the continuum to the TiO band, along with an increase in the angular diameter, may be a consequence of the strong correlation between the estimates of θ_{LD} and u.

2. An empirical relation, known as the Barnes–Evans relation, between the angular diameter, the stellar magnitude V, and the color index (V − R) in the UBVRI system, was calibrated on 76 stars for which there were direct measurements of angular sizes.[22] For this the published values of the angular diameters obtained by various methods (including the method of speckle interferometry) were reduced from θ_{UD} or θ_{LD} with u = 1 to the values with the appropraite u using Eq. (5). Reduction of speckle-interferometric measurements through Eq. (5) (or any other) instead of Eq. (19) seems unfounded to us.

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