Ukrainian Mathematical Journal
August 1993, Volume 45, Issue 8, pp 1215-1220

Convergence near a point and the Arzela-Ascoli-type theorems

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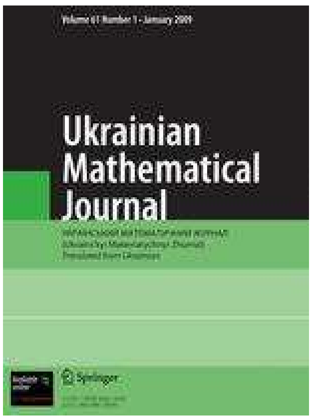
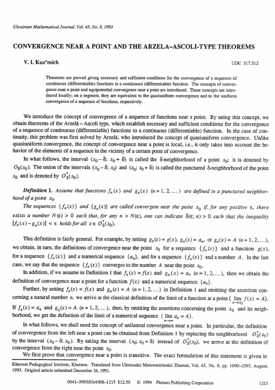
Abstract

Theorems are proved giving necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. The concepts of convergence near a point and equipotential convergence near a point are introduced. These concepts are introduced locally; on a segment, they are equivalent to the quasiuniform convergence and to the uniform convergence of a sequence of functions, respectively.

Translated from Ukrainskii Matemahcheskii Zhurnal, Vol. 45, No. 8, pp. 1090–1095, August, 1993.




Ukrainian Mathematical Journal, Vol. 45, No. 8, 1993

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Other actions

Supplementary Material (0)

References (1)

About this Article

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UDC 517.512

Theorems are proved giving necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. The concepts of convergence near a point and equipotential convergence near a point are introduced. These concepts are introduced locally; on a segment, they are equivalent to the quasiuniform convergence and to the uniform convergence of a sequence of functions, respectively.

We introduce the concept of convergence of a sequence of functions near a point. By using this concept, we obtain theorems of the Arzelá – Ascoli type, which establish necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. In the case of continuity, this problem was first solved by Arzelá, who introduced the concept of quasiuniform convergence. Unlike quasiuniform convergence, the concept of convergence near a point is local, i.e., it only takes into account the behavior of the elements of a sequence in the vicinity of a certain point of convergence.

In what follows, the interval $(x_0-\delta;\ x_0+\delta)$ is called the δ-neighborhood of a point x_0; it is denoted by $O_\delta(x_0)$. The union of the intervals $(x_0-\delta;\ x_0)$ and $(x_0;\ x_0+\delta)$ is called the punctured δ-neighborhood of the point x_0 and is denoted by $O^*_\delta(x_0)$.

***Definition* 1.** *Assume that functions $f_n(x)$ and $g_n(x)$ $(n = 1, 2, \ldots)$ are defined in a punctured neighborhood of a point x_0.*

The sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are called convergent near the point x_0 if, for any positive ε, there exists a number $N(\varepsilon) > 0$ such that, for any $n > N(\varepsilon)$, one can indicate $\delta(\varepsilon; n) > 0$ such that the inequality $|f_n(x) - g_n(x)| < \varepsilon$ holds for all $x \in O^*_\delta(x_0)$.

This definition is fairly general. For example, by setting $g_n(x) = g(x)$, $g_n(x) = a_n$, or $g_n(x) = A$ $(n = 1, 2, \ldots)$, we obtain, in turn, the definitions of convergence near the point x_0 for a sequence $\{f_n(x)\}$ and a function $g(x)$, for a sequence $\{f_n(x)\}$ and a numerical sequence $\{a_n\}$, and for a sequence $\{f_n(x)\}$ and a number A. In the last case, we say that the sequence $\{f_n(x)\}$ converges to the number A near the point x_0.

In addition, if we assume in Definition 1 that $f_n(x) = f(x)$ and $g_n(x) = a_n$ $(n = 1, 2, \ldots)$, then we obtain the definition of convergence near a point for a function $f(x)$ and a numerical sequence $\{a_n\}$.

Further, by setting $f_n(x) = f(x)$ and $g_n(x) = A$ $(n = 1, 2, \ldots)$ in Definition 1 and omitting the assertion concerning a natural number n, we arrive at the classical definition of the limit of a function at a point ($\lim\limits_{x\to x_0} f(x) = A$). If $f_n(x) = a_n$ and $g_n(x) = A$ $(n = 1, 2, \ldots)$, then, by omitting the assertions concerning the point x_0 and its neighborhood, we get the definition of the limit of a numerical sequence ($\lim\limits_{n\to\infty} a_n = A$).

In what follows, we shall need the concept of unilateral convergence near a point. In particular, the definition of convergence from the left near a point can be obtained from Definition 1 by replacing the neighborhood $O^*_\delta(x_0)$ by the interval $(x_0-\delta;\ x_0)$. By taking the interval $(x_0;\ x_0+\delta)$ instead of $O^*_\delta(x_0)$, we arrive at the definition of convergence from the right near the point x_0.

We first prove that convergence near a point is transitive. The exact formulation of this statement is given in

Kherson Pedagogical Institute, Kherson. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 45, No. 8, pp. 1090–1095, August, 1993. Original article submitted December 16, 1991.

the following lemma:

Lemma. *If the sequences* $\{f_n(x)\}$ *and* $\{g_n(x)\}$ *converge near a point* x_0, *and the same is true for the sequences* $\{g_n(x)\}$ *and* $\{h_n(x)\}$, *then the sequences* $\{f_n(x)\}$ *and* $\{h_n(x)\}$ *also converge near this point.*

Proof. Since the sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ converge near the point x_0, we conclude that, for any $\varepsilon > 0$, one can find a number $N_1(\varepsilon) > 0$ such that, for any $n > N_1(\varepsilon)$, there exists a punctured $\delta_1(\varepsilon; n)$-neighborhood of the point x_0 such that the inequality $|f_n(x) - g_n(x)| < \varepsilon$ holds at every point of this neighborhood.

Similarly, the convergence of the sequences $\{g_n(x)\}$ and $\{h_n(x)\}$ near the point x_0 implies, for the same ε, the existence of a number $N_2(\varepsilon) > 0$ such that, for any $n > N_2(\varepsilon)$, a punctured $\delta_2(\varepsilon; n)$-neighborhood of the point x_0 can be found at each point of which the inequality $|g_n(x) - h_n(x)| < \varepsilon$ holds.

By setting $N(\varepsilon) = \max\{N_1, N_2\}$ and $\delta(\varepsilon; n) = \min\{\delta_1, \delta_2\}$, we conclude that the inequality

$$|f_n(x) - h_n(x)| \le |f_n(x) - g_n(x)| + |g_n(x) - h_n(x)| < 2\varepsilon$$

holds for any number $n > N(\varepsilon)$ at every point of the punctured $\delta(\varepsilon; n)$-neighborhood of the point x_0, and this means that the sequences $\{f_n(x)\}$ and $\{h_n(x)\}$ converge near the point x_0. Note that this lemma remains valid for various special cases of convergence near a point.

The following theorem demonstrates that the limiting properties of the elements of a sequence of functions are preserved:

Theorem 1. *Let the functions* $f(x)$ *and* $f_n(x)$ $(n = 1, 2, \ldots)$ *be defined in a punctured neighborhood of a point* x_0. *Assume that the following conditions hold:*

(i) $\lim\limits_{x \to x_0} f_n(x) = a_n \quad (n = 1, 2, \ldots)$;

(ii) $\lim\limits_{n \to \infty} a_n = A$.

In order that the function $f(x)$ *have the limit* A *at the point* x_0, *it is necessary and sufficient that the sequence* $\{f_n(x)\}$ *and the function* $f(x)$ *be convergent near this point.*

Proof. Condition (i) means that, for any $\varepsilon > 0$ and natural n, there exists a number $\delta(\varepsilon; n) > 0$ such that the inequality $|f_n(x) - a_n| < \varepsilon$ holds for all $x \in O^*_\delta(x_0)$, i.e., the sequences $\{f_n(x)\}$ and $\{a_n\}$ converge near the point x_0. As mentioned above, condition (ii) means that the sequence $\{a_n\}$ and the number A converge near an arbitrary point (and, in particular, near the point x_0). By virtue of the lemma, these conditions imply the convergence of the sequence $\{f_n(x)\}$ and the number A near the point x_0.

Necessity. Let $\lim\limits_{x \to x_0} f(x) = A$. Then the function $f(x)$ and the number A converge near the point x_0. Consequently, by the transitivity property, the sequence $\{f_n(x)\}$ and the function $f(x)$ converge near this point.

Sufficiency. Assume that the sequence $\{f_n(x)\}$ and the function $f(x)$ converge near the point x_0. Then, in view of transitivity, the function $f(x)$ and the number A converge near this point, i.e., $\lim\limits_{x \to x_0} f(x) = A$.

This theorem is local but can be extended to an arbitrary interval.