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Ukrainian Mathematical Journal  
August 1993, Volume 45, Issue 8, pp 1215-1220

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## Abstract

Theorems are proved giving necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. The concepts of convergence near a point and equipotential convergence near a point are introduced. These concepts are introduced locally; on a segment, they are equivalent to the quasiuniform convergence and to the uniform convergence of a sequence of functions, respectively.

*Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 45, No. 8, pp. 1090–1095, August, 1993.*



Ukrainian Mathematical Journal, Vol. 45, No. 8, 1993

**CONVERGENCE NEAR A POINT AND THE ARZELA-ASCOLI-TYPE THEOREMS**

V. I. Kuz'mich

There are proved giving necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. The concepts of convergence near a point and equipotential convergence near a point are introduced. These concepts are introduced locally; on a segment, they are equivalent to the quasiuniform convergence and to the uniform convergence of a sequence of functions, respectively.

We introduce the concept of convergence of a sequence of functions near a point. By using this concept, we obtain necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. In the case of equipotential convergence, the concept of convergence near a point is used. It is noted that this concept is the basis of the definition of a sequence in the theory of a certain point of convergence.

In what follows, the interval  $C_0 = [0, 1]$  is called the neighborhood of a point  $x_0$  if it is defined by  $C_0(x_0)$ . The union of the intervals  $C_0 = [0, x_0]$  and  $C_0(x_0) = [x_0, 1]$  is called the punctured neighborhood of the point  $x_0$  and denoted by  $C_0^*(x_0)$ .

**Definition 1.** Assume that functions  $f_n(x)$  and  $f(x)$  ( $n = 1, 2, \dots$ ) are defined in a punctured neighborhood of a point  $x_0$ .

The sequence  $\{f_n(x)\}$  and  $\{f(x)\}$  are called *convergent near the point  $x_0$*  if, for any positive  $\epsilon$ , there exists a number  $N(\epsilon) > 0$  such that, for any  $n > N(\epsilon)$ , one can indicate  $\delta(\epsilon, n) > 0$  such that the inequality  $|f_n(x) - f(x)| < \epsilon$  holds for all  $x \in C_0^*(x_0)$ .

This definition is fairly general. For example, by setting  $f_n(x) = x^n$ ,  $f(x) = x$ , or  $f_n(x) = \sin nx$  ( $n = 1, 2, \dots$ ), we obtain, in turn, the definition of convergence near the point  $x_0$  for a sequence  $\{f_n(x)\}$  and a function  $f(x)$ , that is, convergence  $\{f_n(x)\}$  and uniform convergence  $f(x)$ , and the sequence  $\{f_n(x)\}$  and number  $x_0$ . In the last case, we see that the sequence  $\{f_n(x)\}$  converges to the number  $A$  near the point  $x_0$ .

In addition, for reasons in Definition 1, the  $\{f_n(x)\}$  and  $\{f(x)\}$  will be called *convergent near the point  $x_0$* , that is, the definition of convergence near a point for a function  $f(x)$  and a numerical sequence  $\{a_n\}$ .

Further, by setting  $f_n(x) = f(x)$  and  $f(x) = f(x)$  ( $n = 1, 2, \dots$ ), in Definition 1, and taking the notation concerning a natural number  $n$ , we arrive at the classical definition of the limit of a function at a point  $\lim_{x \rightarrow x_0} f(x) = A$ .

If  $C_0^*(x_0) = [0, 1]$  and  $C_0^*(x_0) = [0, 1]$ , then, by setting the conditions concerning the point  $x_0$  and its neighborhood, we get the definition of the limit of a numerical sequence  $\lim_{n \rightarrow \infty} a_n = A$ .

In what follows, we shall use the concept of uniform convergence near a point. In particular, the definition of convergence near a point will have a precise sense obtained from Definition 1 by replacing the neighborhood  $C_0^*(x_0)$  by the interval  $C_0 = [0, 1]$ . By taking the interval  $C_0 = [0, 1]$  instead of  $C_0^*(x_0)$ , we arrive at the definition of convergence near a point for a function  $f(x)$  and a numerical sequence  $\{a_n\}$ .

It should be noted that the concept of convergence near a point is local. The main introduction of this concept is given in the Introduction of the present paper.

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## CONVERGENCE NEAR A POINT AND THE ARZELÀ-ASCOLI-TYPE THEOREMS

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UDC 517.512

Theorems are proved giving necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. The concepts of convergence near a point and equipotential convergence near a point are introduced. These concepts are introduced locally; on a segment, they are equivalent to the quasiuniform convergence and to the uniform convergence of a sequence of functions, respectively.

We introduce the concept of convergence of a sequence of functions near a point. By using this concept, we obtain theorems of the Arzelà–Ascoli type, which establish necessary and sufficient conditions for the convergence of a sequence of continuous (differentiable) functions to a continuous (differentiable) function. In the case of continuity, this problem was first solved by Arzelà, who introduced the concept of quasiuniform convergence. Unlike quasiuniform convergence, the concept of convergence near a point is local, i.e., it only takes into account the behavior of the elements of a sequence in the vicinity of a certain point of convergence.

In what follows, the interval  $(x_0 - \delta; x_0 + \delta)$  is called the  $\delta$ -neighborhood of a point  $x_0$ ; it is denoted by  $O_\delta(x_0)$ . The union of the intervals  $(x_0 - \delta; x_0)$  and  $(x_0; x_0 + \delta)$  is called the punctured  $\delta$ -neighborhood of the point  $x_0$  and is denoted by  $O_\delta^*(x_0)$ .

**Definition 1.** Assume that functions  $f_n(x)$  and  $g_n(x)$  ( $n = 1, 2, \dots$ ) are defined in a punctured neighborhood of a point  $x_0$ .

The sequences  $\{f_n(x)\}$  and  $\{g_n(x)\}$  are called convergent near the point  $x_0$  if, for any positive  $\varepsilon$ , there exists a number  $N(\varepsilon) > 0$  such that, for any  $n > N(\varepsilon)$ , one can indicate  $\delta(\varepsilon; n) > 0$  such that the inequality  $|f_n(x) - g_n(x)| < \varepsilon$  holds for all  $x \in O_\delta^*(x_0)$ .

This definition is fairly general. For example, by setting  $g_n(x) = g(x)$ ,  $g_n(x) = a_n$ , or  $g_n(x) = A$  ( $n = 1, 2, \dots$ ), we obtain, in turn, the definitions of convergence near the point  $x_0$  for a sequence  $\{f_n(x)\}$  and a function  $g(x)$ , for a sequence  $\{f_n(x)\}$  and a numerical sequence  $\{a_n\}$ , and for a sequence  $\{f_n(x)\}$  and a number  $A$ . In the last case, we say that the sequence  $\{f_n(x)\}$  converges to the number  $A$  near the point  $x_0$ .

In addition, if we assume in Definition 1 that  $f_n(x) = f(x)$  and  $g_n(x) = a_n$  ( $n = 1, 2, \dots$ ), then we obtain the definition of convergence near a point for a function  $f(x)$  and a numerical sequence  $\{a_n\}$ .

Further, by setting  $f_n(x) = f(x)$  and  $g_n(x) = A$  ( $n = 1, 2, \dots$ ) in Definition 1 and omitting the assertion concerning a natural number  $n$ , we arrive at the classical definition of the limit of a function at a point ( $\lim_{x \rightarrow x_0} f(x) = A$ ). If  $f_n(x) = a_n$  and  $g_n(x) = A$  ( $n = 1, 2, \dots$ ), then, by omitting the assertions concerning the point  $x_0$  and its neighborhood, we get the definition of the limit of a numerical sequence ( $\lim_{n \rightarrow \infty} a_n = A$ ).

In what follows, we shall need the concept of unilateral convergence near a point. In particular, the definition of convergence from the left near a point can be obtained from Definition 1 by replacing the neighborhood  $O_\delta^*(x_0)$  by the interval  $(x_0 - \delta; x_0)$ . By taking the interval  $(x_0; x_0 + \delta)$  instead of  $O_\delta^*(x_0)$ , we arrive at the definition of convergence from the right near the point  $x_0$ .

We first prove that convergence near a point is transitive. The exact formulation of this statement is given in

Kherson Pedagogical Institute, Kherson. Translated from *Ukrainskii Matematicheskii Zhurnal*, Vol. 45, No. 8, pp. 1090–1095, August, 1993. Original article submitted December 16, 1991.

the following lemma:

**Lemma.** *If the sequences  $\{f_n(x)\}$  and  $\{g_n(x)\}$  converge near a point  $x_0$ , and the same is true for the sequences  $\{g_n(x)\}$  and  $\{h_n(x)\}$ , then the sequences  $\{f_n(x)\}$  and  $\{h_n(x)\}$  also converge near this point.*

**Proof.** Since the sequences  $\{f_n(x)\}$  and  $\{g_n(x)\}$  converge near the point  $x_0$ , we conclude that, for any  $\varepsilon > 0$ , one can find a number  $N_1(\varepsilon) > 0$  such that, for any  $n > N_1(\varepsilon)$ , there exists a punctured  $\delta_1(\varepsilon; n)$ -neighborhood of the point  $x_0$  such that the inequality  $|f_n(x) - g_n(x)| < \varepsilon$  holds at every point of this neighborhood.

Similarly, the convergence of the sequences  $\{g_n(x)\}$  and  $\{h_n(x)\}$  near the point  $x_0$  implies, for the same  $\varepsilon$ , the existence of a number  $N_2(\varepsilon) > 0$  such that, for any  $n > N_2(\varepsilon)$ , a punctured  $\delta_2(\varepsilon; n)$ -neighborhood of the point  $x_0$  can be found at each point of which the inequality  $|g_n(x) - h_n(x)| < \varepsilon$  holds.

By setting  $N(\varepsilon) = \max\{N_1, N_2\}$  and  $\delta(\varepsilon; n) = \min\{\delta_1, \delta_2\}$ , we conclude that the inequality

$$|f_n(x) - h_n(x)| \leq |f_n(x) - g_n(x)| + |g_n(x) - h_n(x)| < 2\varepsilon$$

holds for any number  $n > N(\varepsilon)$  at every point of the punctured  $\delta(\varepsilon; n)$ -neighborhood of the point  $x_0$ , and this means that the sequences  $\{f_n(x)\}$  and  $\{h_n(x)\}$  converge near the point  $x_0$ . Note that this lemma remains valid for various special cases of convergence near a point.

The following theorem demonstrates that the limiting properties of the elements of a sequence of functions are preserved:

**Theorem 1.** *Let the functions  $f(x)$  and  $f_n(x)$  ( $n = 1, 2, \dots$ ) be defined in a punctured neighborhood of a point  $x_0$ . Assume that the following conditions hold:*

$$(i) \quad \lim_{x \rightarrow x_0} f_n(x) = a_n \quad (n = 1, 2, \dots);$$

$$(ii) \quad \lim_{n \rightarrow \infty} a_n = A.$$

*In order that the function  $f(x)$  have the limit  $A$  at the point  $x_0$ , it is necessary and sufficient that the sequence  $\{f_n(x)\}$  and the function  $f(x)$  be convergent near this point.*

**Proof.** Condition (i) means that, for any  $\varepsilon > 0$  and natural  $n$ , there exists a number  $\delta(\varepsilon; n) > 0$  such that the inequality  $|f_n(x) - a_n| < \varepsilon$  holds for all  $x \in O_{\delta}^*(x_0)$ , i.e., the sequences  $\{f_n(x)\}$  and  $\{a_n\}$  converge near the point  $x_0$ . As mentioned above, condition (ii) means that the sequence  $\{a_n\}$  and the number  $A$  converge near an arbitrary point (and, in particular, near the point  $x_0$ ). By virtue of the lemma, these conditions imply the convergence of the sequence  $\{f_n(x)\}$  and the number  $A$  near the point  $x_0$ .

**Necessity.** Let  $\lim_{x \rightarrow x_0} f(x) = A$ . Then the function  $f(x)$  and the number  $A$  converge near the point  $x_0$ . Consequently, by the transitivity property, the sequence  $\{f_n(x)\}$  and the function  $f(x)$  converge near this point.

**Sufficiency.** Assume that the sequence  $\{f_n(x)\}$  and the function  $f(x)$  converge near the point  $x_0$ . Then, in view of transitivity, the function  $f(x)$  and the number  $A$  converge near this point, i.e.,  $\lim_{x \rightarrow x_0} f(x) = A$ .

This theorem is local but can be extended to an arbitrary interval.