DOI 10.1007/s11253-019-01656-1
Ukrainian Mathematical Journal, Vol. 71, No. 3, August, 2019 (Ukrainian Original Vol. 71, No. 3, March, 2019)

GEOMETRIC PROPERTIES OF METRIC SPACES

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UDC 515.124.4

We study some problems of geometrization of arbitrary metric spaces. In particular, we analyze the notions of straight and flat placements of points in these spaces. We continue the investigations of Kagan devoted to the detailed analysis of the notion of rectilinearity based on four groups of postulates. Our results are based on the notion of angular characteristics of three points of the space proposed by Alexandrov. We establish the conditions under which the set of points of an arbitrary metric space satisfies all five postulates of the first group of Kagan's placement postulates. The relationship between the rectilinear and flat placements of points in the metric space is investigated. Examples of placements of this kind based on linear functions in some classical spaces are presented. The presented results are obtained without using the property of completeness of the space and can be used for the discrete calculations and structuring of specific metric spaces.

1. Introduction

The present paper is devoted to the problems of "geometrization" of an arbitrary metric space, i.e., to the introduction of notions similar to the principal classical geometric notions in these spaces: line, straight line, angle, and plane. As a specific feature of the present paper, we can mention the fact that we do not use the notion of limit transition in analyzing the posed problems and, hence, the notion of completeness of the space, which necessarily appears in the construction of a complete analog of the Euclidean geometry in an arbitrary metric space. In our opinion, this approach enables one to use the accumulated results in finite metric spaces.

The notion of metric space is one of the central notions of mathematics. Parallel with metric spaces, the researchers also perform extensive investigations of their special classes and modifications with numerous applications in various fields of contemporary mathematics. In this connection, we especially mention ultrametric or non-Archimedean spaces (e.g., in [1], the notion of ultrametric is considered for free groups) and fuzzy metric spaces (see, e.g., [2], where a fuzzy metrization of the space of probability measures is constructed).

The unique numerical characteristic of an arbitrary metric space (X, ρ) is the distance $\rho(x, y)$ between arbitrary elements (points) x and y of this space. This partially explains significant difficulties encountered in its geometrization because the introduction of analogs of the principal geometric notions of the Euclidean geometry (straight line, angle, and plane) inevitably requires the property of completeness of the space.

In our opinion, in any metric space, in some cases (e.g., in the case of a space with finite or countable number of points), it is possible to introduce the notions of angle, parallelism, and perpendicularity without using the requirement of completeness of the space if we do not try to create a complete analog of the Euclidean geometry. In a similar way, Kagan considered the notion of "rectilinear placement" of points in a metric space and "rectilinear image." Following Aleksandrov [3, p. 36], as a characteristic of these notions and properties, it is possible to take one of the numerical characteristics of plane angle in the Euclidean geometry. In this case, we can introduce the notion of "flat placement" of points of metric space as an analog of a plane in the Euclidean geometry.

In [4, pp. 260–297], Kagan constructed the axiomatic theory of Euclidean straight line and proposed four groups of postulates: the placement postulates I_{1-5}, the structure postulates II_{1-3}, the congruence postulates III_{1-7},

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 71, No. 3, pp. 382–399, March, 2019. Original article submitted May 16, 2018.

the Archimedean postulate IV_1, and the Cantor postulate IV_2. In [5, p. 29], for the investigation of the notion of rectilinearity in an arbitrary metric space, we introduced the notion of angle formed by three points of the space (as an ordered triple of these points) and the notion of angular characteristic. The indicated characteristic is based on the cosine formula. In [6, pp. 11, 12; 7, pp. 42, 43], on the basis of the notions of angle and angular characteristic, we introduced the notion of flat placement of points of an arbitrary metric space by equating an analog of the determinant of the Gram matrix of a system of unit vectors to zero.

In the present paper, we prove some statements announced in [6], introduce the notion of rectilinear ordering of points in a metric space, and show that, under the condition of rectilinear ordering of points of a certain set in an arbitrary metric space, it satisfies the Kagan postulates I_{1-5}.

The notion of rectilinear ordering of points in the metric space was studied in detail in [4]. In the form used in the present paper, this notion is encountered in [8, p. 527].

The aim of the present paper is to develop an algorithm for the construction of ordinary geometric objects and notions of the Euclidean and non-Euclidean geometries in a metric space, which would enable us to introduce a structure in this space.

2. Preliminary Information

We now present definitions introduced in our previous works with slight modifications performed for better understanding of our subsequent reasoning.

In what follows, we assume that all points of a space are different, i.e., consider only positive values of the metric of the space. We say that a collection of three points $a,\ b,$ and c of the space forms a triangle and denote it by $\triangle\ (a,b,c)$. These points are called vertices and the pairs of points $(a,b),\ (b,c),$ and (a,c) are called the sides of the triangle.

Definition 1. *Let $a,\ b,$ and c be arbitrary points of a metric space (X,ρ). An ordered triple (a,b,c) of these points is called an angle with vertex at the point b and denoted by $\angle(a,b,c)$. Moreover, the pairs of points (a,b) and (b,c) are called the sides of the angle (see [5, p. 28]).*

Definition 2. *Let $a,\ b,$ and c be arbitrary points of the metric space (X,ρ). The characteristic of the angle $\angle(a,b,c)$ or the angular characteristic is defined as a real number $\varphi(a,b,c)$ given by the formula*

$$\varphi(a,b,c)=\frac{\rho^2(a,b)+\rho^2(b,c)-\rho^2(a,c)}{2\rho(a,b)\rho(b,c)} \tag{1}$$

(see [3, p. 36; 5, p. 29]).

A metric space (X,ρ) with the notions of angle and its characteristic introduced by Definitions 1 and 2, respectively, is called a metric space with angular characteristic and denoted by Π.

Definition 3. *We say that the points $a,\ b,$ and c of the space Π are rectilinearly placed if the equality*

$$\varphi^2(a,b,c)=1 \tag{2}$$

holds for at least one of these points (e.g., for the point b) (see [5, p. 29]).

Definition 4. *We say that a set of points of the space Π is rectilinearly placed if any three points of this set are rectilinearly placed (see [7, p. 527]).*