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Analytical and geometric interpretation of the flat arrangement of points by means of metric geometry in the study of metric spaces

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Abstract. When studying metric spaces, students of higher education often have difficulties with understanding the basic concepts and properties of these spaces. This, to a large extent, is a consequence of the significant level of formalization of such concepts on the one hand, and the preservation of the corresponding formulations and names familiar to students from a school mathematics course. To overcome these difficulties, it is advisable to use methods of geometric interpretation and visualization of these properties. At the same time, it is appropriate to use elements of metric geometry. Its methods make it possible to interpret the geometric features of the mutual placement of points of metric space in Cartesian (rectangular) coordinate systems, which are familiar to students of higher education. Moreover, it becomes possible to visualize these features with the help of graphic editors, since they, as a rule, use numerical values of the coordinates of points to visualize them. Based on the definition of an angle as an ordered trio of points of an arbitrary metric space, and the angular characteristic of this angle, the fact of the flat placement of four points of a non-Euclidean metric space is established, and examples of digital visualization of this arrangement using the dynamic geometric environment GeoGebra 3D are given.

1. Introduction

The assimilation of basic concepts of metric spaces by students of higher education causes them certain difficulties, which are associated with a significant level of formalization of these concepts. In some cases, the geometric properties of non-Euclidean metric spaces can directly contradict the corresponding concepts and properties of classical Euclidean spaces. These contradictions, to a large extent, can be explained to the acquirers by constructing a geometric interpretation of a separate concept in a specific metric space. This paper proposes the use of elements of metric geometry to construct such interpretations. This makes it possible to make not only a graphic interpretation of a separate concept of a specific metric space, but also its visualization in classical Cartesian (rectangular) coordinate systems, using digital technologies.

Among the studies devoted to the issues of geometrization of metric spaces, one should single out the fundamental works [1–4], in which the main provisions and the latest research on metric geometry are outlined. Among domestic works devoted to various issues of geometrization of metric spaces, the following works can be noted: [5, 6]. The issues of geometric interpretation and visualization of geometric properties of metric spaces in the course of higher mathematics were considered in works [7–10]. The issue of introducing elements of the theory of metric spaces into the school mathematics course and extracurricular work in mathematics was considered in works [11, 12].

The paper considers an example of flat placement of points in metric space. This flat placement is demonstrated using the dynamic geometry environment GeoGebra 3D.

The purpose of the article is to demonstrate the possibility of applying the elements of metric geometry to the construction of geometric interpretation and digital visualization of the main concepts of the theory of metric spaces, with the aim of improving their assimilation by students of higher education.

2. Preliminary information

At the beginning of the work, we will present some of the main definitions and facts of the theory of metric spaces. Basic of them are the concepts of space, points of space, distance between points of space.

Suppose that in the set X of elements x, according to a certain rule ρ, any two different elements x_1 and x_2 of this set can be matched with a single real number $\rho(x_1, x_2)$ so that the following conditions are met:

1) $\rho(x_1, x_2) > 0$;

2) $\rho(x_1, x_2) = \rho(x_2, x_1)$;

3) $\rho(x_1, x_2)\rho(x_1, x_3) + \rho(x_2, x_3$, for any element x_3 of the set, then such a rule ρ is called the metric of the set X, the set itself is called the metric space with the metric ρ and denoted by (X, ρ), the numerical value $\rho(x_1, x_2)$ is the distance between the elements x_1 and x_2, and the elements themselves are points of the metric space. The method of setting the space metric determines its properties [13–15]. In the future, the distance between the points x_i and x_j will be briefly denoted by ρ_{ij}. The simplest examples of metric spaces are the one-dimensional (R^1), two-dimensional (R^2) and three-dimensional (R^3) Euclidean spaces, with which students of higher education are familiar from a school course in mathematics.

An example of a metric space is the space $C_{[0;1]}$ – the space of real functions continuous on the segment [0;1]. In this space, the distance between the functions $f(x)$ and $g(x)$ is given by the formula [16, 17]:

$$\rho(f, g) = \max_{x\in[0;1]} \big|f(x) - g(x)\big|. \tag{1}$$

If in condition 3) the inequality turns into equality, then it is said that the points x_1, x_2, x_3 are located in a straight line in the space are located in a straight line in the space X [18, 19]. A certain set of points of a metric space will be called rectilinearly placed in this space if any three of its points are rectilinearly placed. The definition of flat placement of points uses the concept of an angle formed by three points x_1, x_2, x_3 of the metric space (X, ρ). An ordered triple of points (x_1, x_2, x_3) will be called an angle, in which the point x_2 is called the vertex of the angle, and the pairs of points (x_1, x_2) and (x_2, x_3) are the sides of the angle. To denote it, you can use the classic angle notation: $\angle(x_1, x_2, x_3)$ will be called the real number $\varphi(x_1, x_2, x_3)$, which is found according to the formula [20]:

$$\varphi(x_1, x_2, x_3) = \frac{\rho^2(x_1, x_2) + \rho^2(x_2, x_3) - \rho^2(x_1, x_3)}{2\rho(x_1, x_2)\rho(x_2, x_3)},$$

or shorter:

$$\varphi_{123} = \frac{\rho_{12}^2 + \rho_{23}^2 - \rho_{13}^2}{2\rho_{12}\rho_{23}}. \tag{2}$$

The definition of the flat arrangement of four different points of the metric space is based on the fact that the volume of the tetrahedron whose vertices are these points is equal to zero. Let

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us say that four different points x_1, x_2, x_3, x_4 of the space (X, ρ) are flatly placement in this space, if the equality holds [10]:

$$\begin{vmatrix} 1 & \varphi_{213} & \varphi_{214} \\ \varphi_{213} & 1 & \varphi_{314} \\ \varphi_{214} & \varphi_{314} & 1 \end{vmatrix} = 1 + 2\varphi_{213}\varphi_{214}\varphi_{314} - \varphi_{213}^2 - \varphi_{214}^2 - \varphi_{314}^2 = 0. \quad (3)$$

A certain set of points of a metric space will be called flatly placement in this space if any four of its points are flatly placement in this space.

3. Main results

We will give an example of modeling the mutual placement of four different points of space $C_{[a;b]}$ using the dynamic geometric environment GeoGebra 3D. To do this, let's establish a certain orientation of the tetrahedron, the vertices of which are these points. We denote by $A(x_A, y_A, z_A)$, $B(x_B, y_B, z_B)$, $C(x_C, y_C, z_C)$, $S(x_S, y_S, z_S)$ the vertices of the tetrahedron. We denote the lengths of the edges of the tetrahedron: $AB = a_1$, $AS = a_2$, $AC = a_3$, $BS = a_4$, $BC = a_5$, $CS = a_6$. We place vertex A in the center of the system of three-dimensional Cartesian (rectangular) coordinates (space R^3), and vertex B - on the positive half-axis of the abscissa (figure 1).

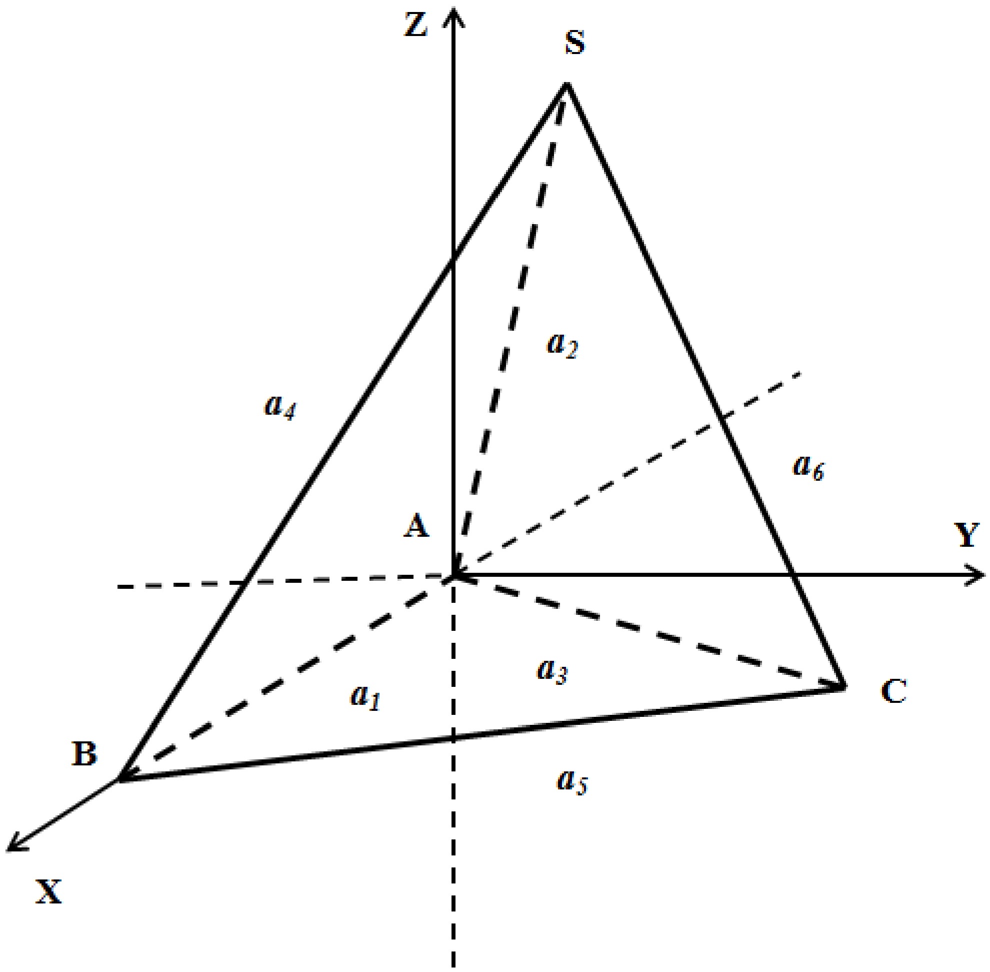

Figure 1. Orientation of the tetrahedron in the R^3 space.

When calculating the coordinates of the vertices of the tetrahedron, we will always choose the ordinate of point C and the applicate of point S as non-negative. The formulas for the coordinates of the vertices of the tetrahedron, with this orientation, will be:

$x_A = 0$; $y_A = 0$; $z_A = 0$.

$x_B = a_1$; $y_B = 0$; $z_B = 0$.

$x_C = \frac{1}{2a_1}(a_1^2 + a_3^2 - a_5^2)$;

$y_C = \frac{1}{2a_1}\sqrt{2(a_1^2a_3^2 + a_1^2a_5^2 + a_3^2a_5^2) - a_1^4 - a_3^4 - a_5^4}$;

$z_C = 0$.

$x_S = \frac{1}{2a_1}(a_1^2 + a_2^2 - a_4^2)$;

$y_S = \frac{2a_1^2a_2^2 + 2a_1^2a_3^2 - 2a_1^2a_6^2 - (a_1^2+a_2^2-a_4^2)(a_1^2+a_3^2-a_5^2)}{2a_1\sqrt{2(a_1^2a_3^2+a_1^2a_5^2+a_3^2a_5^2)-a_1^4-a_3^4-a_5^4}}$;

$$z_S = \sqrt{\frac{a_1^2a_6^2(a_2^2+a_3^2+a_4^2+a_5^2-a_1^2-a_6^2) + a_2^2a_5^2(a_1^2+a_3^2+a_4^2+a_6^2-a_2^2-a_5^2) + a_3^2a_4^2(a_1^2+a_2^2+a_5^2+a_6^2-a_3^2-a_4^2) - a_2^2a_3^2a_6^2 - a_1^2a_3^2a_5^2 - a_1^2a_2^2a_4^2 - a_4^2a_5^2a_6^2}{(a_1+a_3+a_5)(a_3+a_5-a_1)(a_1+a_5-a_3)(a_1+a_3-a_5)}}.$$

This orientation of the tetrahedron is necessary, otherwise the task of constructing a tetrahedron with given edge lengths becomes indeterminate, since there can be 720 of all possible orientations. This can be demonstrated on the calculator, which calculates the value of the square of the volume of the tetrahedron using the Jungius formula. If this value is negative, then the tetrahedron does not exist with this orientation. If it is equal to zero, then all vertices of the tetrahedron lie in the same plane (are flatly placement). Consider an example of a flatly placement of points in the metric space $C_{[0;1]}$ (figure 2).

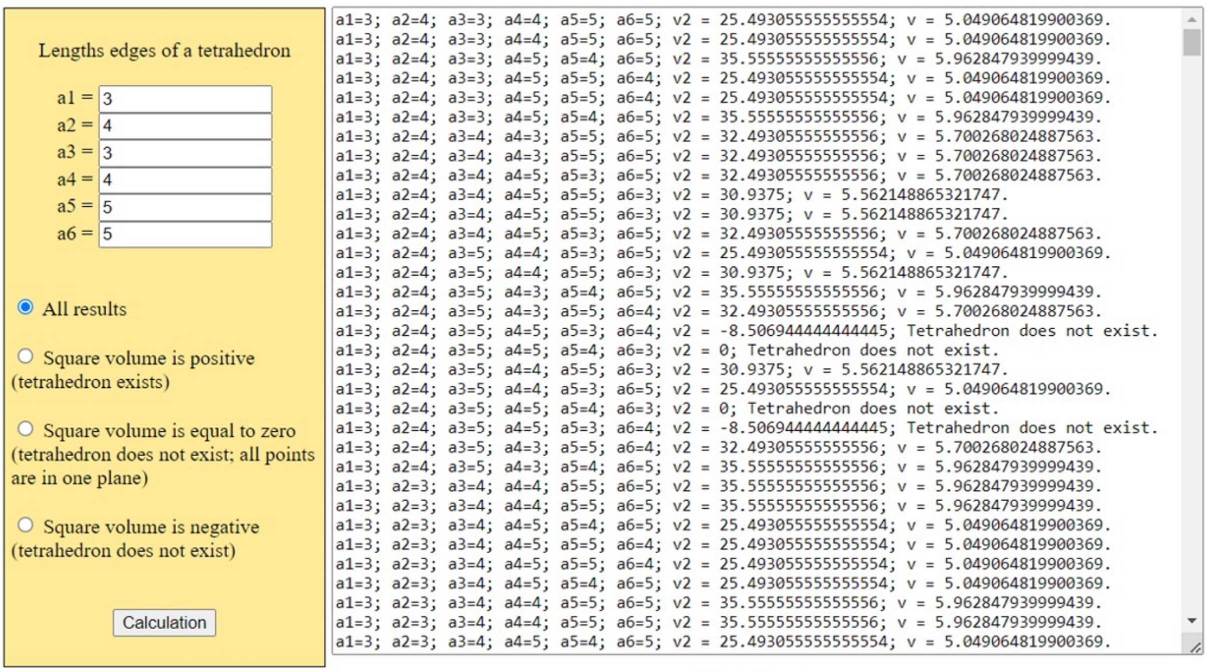

Figure 2. Existence of a tetrahedron with given edge lengths at different orientations.

Example 1. Let's take four points (functions) in the space $C_{[0;1]}$):

$$y_1 = x, y_2 = 0, y_3 = x - 1, y_4 = \frac{2\sqrt{3}}{3}(x - 0,5).$$

Using formula (1), we will find the distances between these points:

$$\rho_{12} = \rho_{13} = \rho_{23} = 1, \rho_{14} = \rho_{24} = \rho_{34} = \frac{\sqrt{3}}{3}.$$

According to formula (2), we find the angular characteristics:

$$\varphi_{142} = \varphi_{143} = \varphi_{243} = -0,5.$$

Substituting these values into the left part of formula (3), we will have:

$$1 + 2(-0,5)(-0,5)(-0,5) - (-0,5)^2 - (-0,5)^2 - (-0,5)^2 = 0.$$

Thus, the points y_1, y_2, y_3, y_4 are flatly placed in the space $C_{[0;1]}$, and no three of these points are rectilinearly placed (there is no distance equal to the sum of the other two).

In Euclid's geometry, in the space R^3, the image of point y_4 is the center of an equilateral triangle with vertices in the images of points y_1, y_2, y_3. That is, the images of the points y_1, y_2, y_3, y_4 of the space $C_{[0;1]}$ are flatly placed in the space R^3.

Modern digital technologies make it possible to visualize certain geometric properties of metric spaces. For example, with the help of the dynamic geometric environment GeoGebra 3D, you can visually make sure that the points y_1, y_2, y_3, y_4 of the space $C_{[0;1]}$, because their images in the space R^3 are flatly placed. To construct points in this environment, the values of their coordinates in the R^3 space are required, which are calculated according to the above formulas. After entering the length values ρ_{12}, ρ_{13}, ρ_{14}, ρ_{23}, ρ_{24}, ρ_{34} from Example 1 into this application, with the appropriate orientation of the tetrahedron, we get the following image of the flatly placement of the points y_1, y_2, y_3, y_4 in the R^3 space (figure 3).

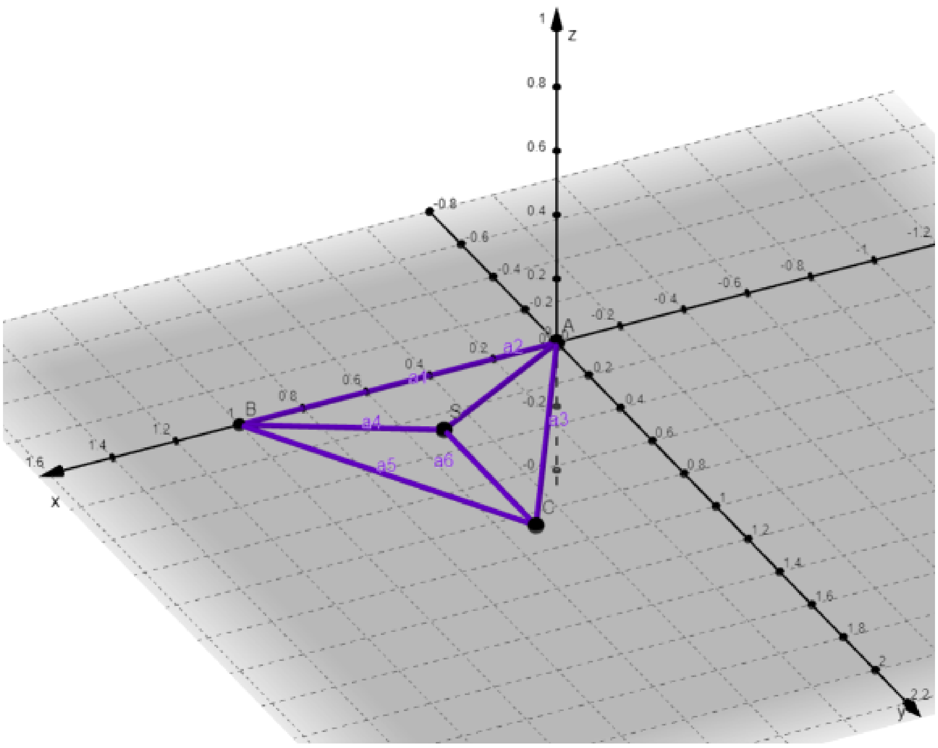

Figure 3. Interpretation of flat placement of points y_1, y_2, y_3, y_4.

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Figure 3 does not give a complete picture of the flat location of points y_1, y_2, y_3, y_4, since it itself lies in a plane. In addition, the point from which the tetrahedron is visible (observation point) is located above the XOY plane. In order to make sure of their flat location, this image can be rotated to some angle in the GeoGebra 3D environment. In other words, you can change the observation point.

By rotating the image so that the viewing point lies in the XOY plane, you can make sure that all four points lie in this plane (figure 4).

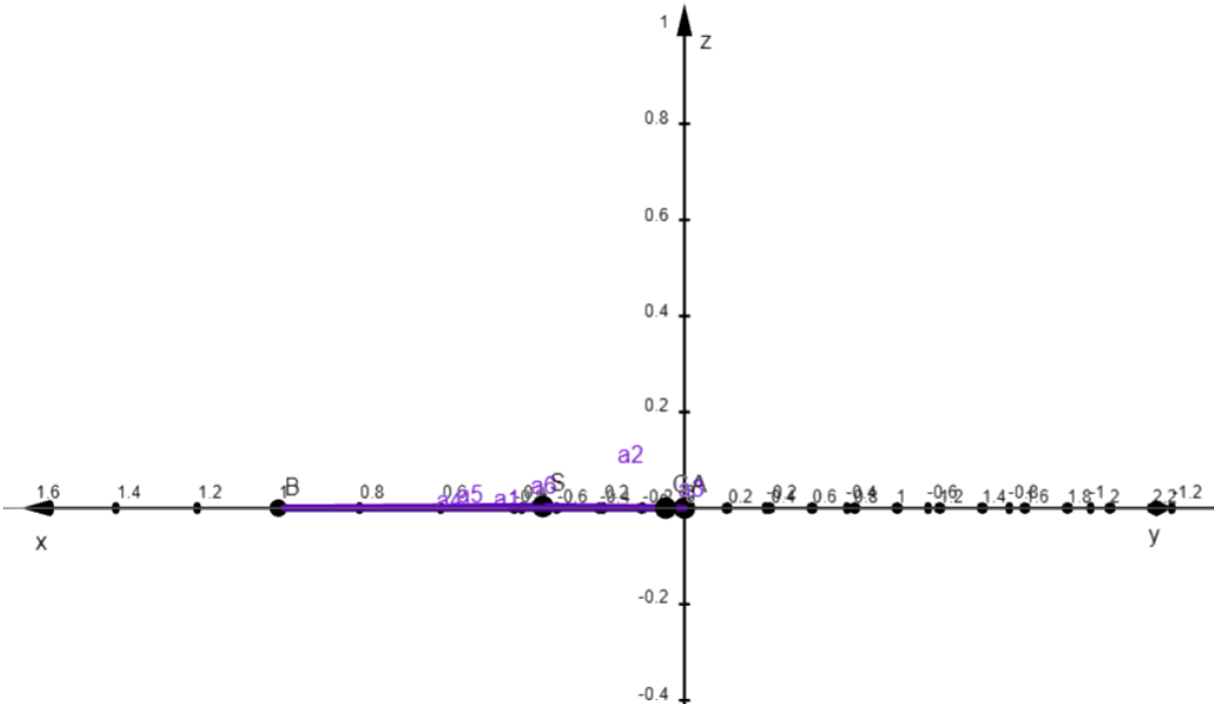

Figure 4. Interpretation of flat placement of points y_1, y_2, y_3, y_4 (view from the XOY plane point).

Digital technologies, in most cases, use approximate values of the coordinates of points, so the visualizations shown in Figures 3 and 4 are illustrative in nature, but sufficiently reflect the nature of mutual placement of points in space.

4. Conclusions

The material presented in this paper testifies to the possibility of using the elements of metric geometry when studying the theory of metric spaces by students of higher education. The analytical apparatus of metric geometry is sufficient for constructing geometric interpretations and digital visualizations of basic concepts and properties of non-Euclidean metric spaces. The use of elements of metric geometry makes it easier for students of higher education to understand those features of non-Euclidean metric spaces that are geometric in nature. The material of this work can be used in various types of non-formal education, introducing it to students of general secondary education who are studying in special classes with in-depth study of mathematics. Its use will make it possible to familiarize students with the simplest elements of non-Euclidean geometry.

The analytical apparatus of metric geometry makes it possible to form a generalized concept of a flat placement of points in an arbitrary metric space. The use of digital technologies, in particular graphic editors, makes it possible to visualize individual features of the mutual placement of points in an arbitrary metric space.

Further research, in our opinion, should be aimed at building analytical and geometric interpretations of parallel and perpendicular placement of points of an arbitrary metric space. This will significantly expand the field of application of metric geometry in the study of metric spaces.

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