Я.Д. Плоткин

ЗАДАЧА КОШИ ДЛЯ ОДНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ ТРЕТЬЕГО ПОРЯДКА

В данной работе методом Римана решается задача Коши для фактаризованого дифференциального уравнения 3-го порядка с частными производными:

$$\begin{cases} \left(\partial_0 + a_{01}\partial_1 + a_{02}\partial_2\right)\left(\partial_0 + a_{11}\partial_1 + a_{12}\partial_2\right)\left(\partial_0 + a_{21}\partial_1 + a_{22}\partial_2\right)u = \left(\dfrac{\lambda}{3}\right)^3 u; & (1) \\ \partial_0^k u\left(0, x_1, x_2\right) = f_k\left(x_1, x_2\right), \quad k = \overline{0, 2}; & (2) \end{cases}$$

где

$$a_{ij} = const, \quad i = \overline{0, 2}; \quad j = \overline{1, 2}; \quad u = u\left(t, x_1, x_2\right);$$
$$\partial_0 u = \frac{\partial u}{\partial t}; \quad \partial_i u = \frac{\partial u}{\partial x_i}; \quad i = \overline{1, 2}.$$

Матрица

$$A = \begin{pmatrix} 1 & a_{01} & a_{02} \\ 1 & a_{11} & a_{12} \\ 1 & a_{21} & a_{22} \end{pmatrix}$$

имеет обратную

$$A^{-1} = \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix}.$$

Функции $f_k\left(x_1, x_2\right), \quad k = \overline{0, 2};$ $\left(2 - k\right)$ раз непрерывно дифференцируемы.

Решение этой задачи имеет вид.

$$u\left(t,x_1,x_2\right)=\frac{1}{3}\Big(f_0\left(x_1-a_{01}t;x_2-a_{02}t\right)+f_0\left(x_1-a_{11}t;x_2-a_{12}t\right)+f_0\left(x_1-a_{21}t;x_2-a_{22}t\right)\Big)-$$

$$-\frac{1}{3}\Bigg(\int_0^{-t}\Big(\alpha_0 f_1+\left(\alpha_1^{(3)}\partial_1+\alpha_2^{(3)}\partial_2\right)f_0\Big)\Big(\left(a_{01}-a_{11}\right)x+x_1-a_{11}t;\left(a_{02}-a_{12}\right)x+x_2-a_{12}t\Big)\mathrm{d}x+$$

$$+\int_0^{-t}\Big(\alpha_0 f_1+\left(\beta_1^{(1)}\partial_1+\beta_2^{(1)}\partial_2\right)f_0\Big)\Big(\left(a_{11}-a_{21}\right)y+x_1-a_{21}t;\left(a_{12}-a_{22}\right)y+x_2-a_{22}t\Big)\mathrm{d}y+$$

$$+\int_0^{-t}\Big(\alpha_0 f_1+\left(\gamma_1^{(2)}\partial_1+\gamma_2^{(2)}\partial_2\right)f_0\Big)\Big(\left(a_{21}-a_{01}\right)z+x_1-a_{01}t;\left(a_{22}-a_{02}\right)z+x_2-a_{02}t\Big)\mathrm{d}z\Bigg)+$$

$$+\frac{1}{3}\Bigg(\int_0^{-t}\mathrm{d}x\int_0^{-t-x}\Big(\Big(v_{xy}-2v_y\left(\alpha_1^{(1)}\partial_1+\alpha_2^{(1)}\partial_2\right)+v_x\left(\beta_1^{(1)}\partial_1+\beta_2^{(1)}\partial_2\right)+v\Big(m_{11}^{(1)}\partial_1^2+2m_{12}^{(1)}\partial_1\partial_2+$$

$$+m_{22}^{(1)}\partial_2^2\Big)\Big)f_0+\Big(-2\alpha_0 v_y+\alpha_0 v_x+v\Big(m_1^{(1)}\partial_1+m_2^{(1)}\partial_2\Big)\Big)f_1+\alpha_0^2 v f_2\Big)\Big(\left(a_{01}-a_{21}\right)x+$$

$$+\left(a_{11}-a_{21}\right)y+x_1-a_{21}t;\left(a_{02}-a_{22}\right)x+\left(a_{12}-a_{22}\right)y+x_2-a_{22}t\Big)\mathrm{d}y+\int_0^{-t}\mathrm{d}y\int_0^{-t-y}\Big(\Big(v_{yz}-$$

$$-2v_z\left(\beta_1^{(2)}\partial_1+\beta_2^{(2)}\partial_2\right)+v_y\left(\gamma_1^{(2)}\partial_1+\gamma_2^{(2)}\partial_2\right)+v\Big(m_{11}^{(2)}\partial_1^2+2m_{12}^{(2)}\partial_1\partial_2+m_{22}^{(2)}\partial_2^2\Big)\Big)f_0+$$

$$+\Big(-2\alpha_0 v_z+\alpha_0 v_y+v\Big(m_1^{(2)}\partial_1+m_2^{(2)}\partial_2\Big)\Big)f_1+\alpha_0^2 v f_2\Big)\Big(\left(a_{11}-a_{01}\right)y+\left(a_{21}-a_{01}\right)z+x_1-a_{01}t;$$

$$\left(a_{12}-a_{02}\right)y+\left(a_{22}-a_{02}\right)z+x_2-a_{02}t\Big)\mathrm{d}z+\int_0^{-t}\mathrm{d}z\int_0^{-t-z}\Big(\Big(v_{zx}-2v_x\left(\gamma_1^{(3)}\partial_1+\gamma_2^{(3)}\partial_2\right)+$$

$$+v_z\left(\alpha_1^{(3)}\partial_1+\alpha_2^{(3)}\partial_2\right)+v\Big(m_{11}^{(3)}\partial_1^2+2m_{12}^{(3)}\partial_1\partial_2+m_{22}^{(3)}\partial_2^2\Big)\Big)f_0+\Big(-2\alpha_0 v_x+\alpha_0 v_z+$$

$$+v\Big(m_1^{(3)}\partial_1+m_2^{(3)}\partial_2\Big)\Big)f_1+\alpha_0^2 v f_2\Big)\Big(\left(a_{01}-a_{11}\right)x+\left(a_{21}-a_{11}\right)z+x_1-a_{11}t;\left(a_{02}-a_{12}\right)x+$$

$$\left(a_{22}-a_{12}\right)z+x_2-a_{12}t\Big)\mathrm{d}x.$$

где $$v=v\left(x,y,z\right)=1+\sum_{k=1}^{\infty}\left(-1\right)^k\cdot\frac{\lambda^k}{\left(k!\right)^3 3^{3k}}\cdot x^k\cdot y^k\cdot z^k \qquad (8)$$

функция Бесселя третьего порядка, нулевого индекса[2];

$$\alpha_0=\frac{1}{b_{00}+b_{01}+b_{02}};\ \alpha_1^{(1)}=\alpha_0\left[\left(b_{01}+b_{02}\right)\left(a_{01}-a_{21}\right)-b_{01}\left(a_{02}-a_{22}\right)\right];$$

$$\alpha_2^{(1)}=\alpha_0\left[\left(b_{01}+b_{02}\right)\left(a_{11}-a_{21}\right)-b_{01}\left(a_{12}-a_{22}\right)\right];$$

$$\beta_1^{(1)}=\alpha_0\left[b_{00}\left(a_{01}-a_{21}\right)-\left(b_{00}+b_{02}\right)\left(a_{02}-a_{22}\right)\right];$$

$$\beta_2^{(1)}=\alpha_0\left[b_{00}\left(a_{11}-a_{21}\right)-\left(b_{00}+b_{02}\right)\left(a_{12}-a_{22}\right)\right];$$

$$m_1^{(1)} = \alpha_0^2\left[\left(-b_{00}+b_{01}+b_{02}\right)\left(a_{01}-a_{21}\right)-\left(b_{00}-b_{01}+b_{02}\right)\left(a_{02}-a_{22}\right)\right];$$

$$m_2^{(1)} = \alpha_0^2\left[\left(-b_{00}+b_{01}+b_{02}\right)\left(a_{11}-a_{21}\right)-\left(b_{00}-b_{01}+b_{02}\right)\left(a_{12}-a_{22}\right)\right];$$

$$m_{11}^{(1)} = \alpha_0^2\Big[-b_{00}\left(b_{01}+b_{02}\right)\left(a_{01}-a_{21}\right)^2-b_{01}\left(b_{00}+b_{02}\right)\left(a_{02}-a_{22}\right)^2+$$
$$+\left(b_{02}\left(b_{01}+b_{02}\right)+b_{00}\left(2b_{01}+b_{02}\right)\right)\left(a_{01}-a_{21}\right)\left(a_{02}-a_{22}\right)\Big];$$

$$m_{12}^{(1)} = \alpha_0^2\Big[-b_{00}\left(b_{01}+b_{02}\right)\left(a_{01}-a_{21}\right)\left(a_{11}-a_{21}\right)-b_{01}\left(b_{00}+b_{02}\right)\times$$
$$\times\left(a_{02}-a_{22}\right)\left(a_{12}-a_{22}\right)+\frac{1}{2}\left(b_{02}\left(b_{01}+b_{02}\right)+b_{00}\left(2b_{01}+b_{02}\right)\right)\times$$
$$\times\left(\left(a_{01}-a_{21}\right)\left(a_{12}-a_{22}\right)+\left(a_{11}-a_{21}\right)\left(a_{02}-a_{22}\right)\right)\Big];$$

$$m_{22}^{(1)} = \alpha_0^2\Big[-b_{00}\left(b_{01}+b_{02}\right)\left(a_{11}-a_{21}\right)^2-b_{01}\left(b_{00}+b_{02}\right)\left(a_{02}-a_{22}\right)^2+$$
$$+\left(b_{02}\left(b_{01}+b_{02}\right)+b_{00}\left(2b_{01}+b_{02}\right)\right)\left(a_{11}-a_{21}\right)\left(a_{12}-a_{22}\right)\Big];$$

$$\beta_1^{(2)} = \alpha_0\left[\left(b_{00}+b_{02}\right)\left(a_{11}-a_{01}\right)-b_{02}\left(a_{12}-a_{02}\right)\right];$$

$$\beta_2^{(2)} = \alpha_0\left[\left(b_{00}+b_{02}\right)\left(a_{21}-a_{01}\right)-b_{02}\left(a_{22}-a_{02}\right)\right];$$

$$\gamma_1^{(2)} = \alpha_0\left[-b_{01}\left(a_{11}-a_{01}\right)+\left(b_{00}+b_{01}\right)\left(a_{12}-a_{02}\right)\right];$$

$$\gamma_2^{(2)} = \alpha_0\left[-b_{01}\left(a_{21}-a_{01}\right)+\left(b_{00}+b_{01}\right)\left(a_{22}-a_{02}\right)\right];$$

$$m_1^{(2)} = \alpha_0^2\left[\left(b_{00}-b_{01}+b_{02}\right)\left(a_{11}-a_{01}\right)+\left(b_{00}+b_{01}-b_{02}\right)\left(a_{12}-a_{02}\right)\right];$$

$$m_2^{(2)} = \alpha_0^2\left[\left(b_{00}-b_{01}+b_{02}\right)\left(a_{21}-a_{01}\right)+\left(b_{00}+b_{01}-b_{02}\right)\left(a_{22}-a_{02}\right)\right];$$

$$m_{11}^{(2)} = \alpha_0^2\Big[-b_{01}\left(b_{00}+b_{02}\right)\left(a_{11}-a_{01}\right)^2-b_{02}\left(b_{00}+b_{01}\right)\left(a_{12}-a_{02}\right)^2+$$
$$+\left(b_{00}^2+2b_{01}\cdot b_{02}+b_{00}\left(b_{01}+b_{02}\right)\right)\left(a_{11}-a_{01}\right)\left(a_{12}-a_{02}\right)\Big];$$

$$m_{12}^{(2)} = \alpha_0^2\Big[-b_{01}\left(b_{00}+b_{02}\right)\left(a_{11}-a_{01}\right)^2-b_{02}\left(b_{00}+b_{01}\right)\times$$
$$\times\left(a_{22}-a_{02}\right)\left(a_{12}-a_{02}\right)+\frac{1}{2}\left(b_{00}^2+2b_{01}\cdot b_{02}+b_{00}\left(b_{01}+b_{02}\right)\right)\times$$
$$\times\left(\left(a_{11}-a_{01}\right)\left(a_{22}-a_{02}\right)+\left(a_{12}-a_{02}\right)\left(a_{21}-a_{01}\right)\right)\Big];$$

$$m_{22}^{(2)} = \alpha_0^2\Big[-b_{01}\left(b_{00}+b_{02}\right)\left(a_{21}-a_{01}\right)^2-b_{02}\left(b_{00}+b_{01}\right)\left(a_{22}-a_{02}\right)^2+$$
$$+\left(b_{00}^2+2b_{01}\cdot b_{02}+b_{00}\left(b_{01}+b_{02}\right)\right)\left(a_{21}-a_{01}\right)\left(a_{22}-a_{02}\right)\Big];$$

$$\gamma_1^{(3)} = \alpha_0 \left[-b_{00} \left(a_{01} - a_{11} \right) + \left(b_{00} + b_{01} \right) \left(a_{02} - a_{12} \right) \right];$$

$$\gamma_2^{(3)} = \alpha_0 \left[-b_{00} \left(a_{21} - a_{11} \right) + \left(b_{00} + b_{01} \right) \left(a_{22} - a_{12} \right) \right];$$

$$\alpha_1^{(3)} = \alpha_0 \left[\left(b_{01} + b_{02} \right) \left(a_{01} - a_{11} \right) - b_{02} \left(a_{02} - a_{12} \right) \right];$$

$$\alpha_2^{(3)} = \alpha_0 \left[\left(b_{01} + b_{02} \right) \left(a_{21} - a_{11} \right) - b_{02} \left(a_{22} - a_{12} \right) \right];$$

$$m_1^{(3)} = \alpha_0^2 \left[\left(-b_{00} + b_{01} + b_{02} \right) \left(a_{01} - a_{11} \right) + \left(b_{00} + b_{01} - b_{02} \right) \left(a_{02} - a_{12} \right) \right];$$

$$m_2^{(3)} = \alpha_0^2 \left[\left(-b_{00} + b_{01} + b_{02} \right) \left(a_{21} - a_{11} \right) + \left(b_{00} + b_{01} - b_{02} \right) \left(a_{22} - a_{12} \right) \right];$$

$$m_{11}^{(3)} = \alpha_0^2 \Big[-b_{00} \left(b_{01} + b_{02} \right) \left(a_{01} - a_{11} \right)^2 - \left(b_{00} + b_{01} \right) b_{02} \left(a_{02} - a_{12} \right)^2 + \\ + \left(b_{01} \left(b_{01} + b_{02} \right) + b_{00} \left(b_{01} + 2b_{02} \right) \right) \left(a_{01} - a_{11} \right) \left(a_{02} - a_{12} \right) \Big];$$

$$m_{12}^{(3)} = \alpha_0^2 \Big[-b_{00} \left(b_{01} + b_{02} \right) \left(a_{01} - a_{11} \right) \left(a_{21} - a_{11} \right) - \left(b_{00} + b_{01} \right) b_{02} \times \\ \times \left(a_{02} - a_{12} \right)^2 + \frac{1}{2} \left(b_{01} \left(b_{01} + b_{02} \right) + b_{00} \left(b_{01} + 2b_{02} \right) \right) \times \\ \times \left(\left(a_{01} - a_{11} \right) \left(a_{22} - a_{12} \right) + \left(a_{21} - a_{11} \right) \left(a_{02} - a_{12} \right) \right) \Big];$$

$$m_{22}^{(3)} = \alpha_0^2 \Big[-b_{00} \left(b_{01} + b_{02} \right) \left(a_{21} - a_{11} \right)^2 - \left(b_{00} + b_{01} \right) b_{02} \left(a_{22} - a_{12} \right)^2 + \\ + \left(b_{01} \left(b_{01} + b_{02} \right) + b_{00} \left(b_{01} + 2b_{02} \right) \right) \left(a_{21} - a_{11} \right) \left(a_{22} - a_{12} \right) \Big];$$

Литература

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